Perturbative corrections to the Ohta-Jasnow-Kawasaki theory of phase-ordering dynamics

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A perturbation expansion is considered about the Ohta-Jasnow-Kawasaki (OJK) theory of phase-ordering dynamics, which is an approximate theory describing the coarsening dynamics of a system with a nonconserved scalar order parameter. In this calculation the nonlinear terms neglected in the OJK equation for the evolution of the auxiliary field are reinstated and treated as a perturbation to the linearized equation. The first order correction term to the pair correlation function is calculated in the large-d limit, and found to be of order $1/d^2$. [S1063-651X(98)07011-1]

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I. INTRODUCTION

When a system is quenched from a high-temperature, homogeneous phase into a two-phase region, domains of the equilibrium phases form and evolve with time. If the order parameter is nonconserved, then the coarsening dynamics are modelled by the time-dependent Ginzburg-Landau (TDGL) equation [1].

In one dimension this system is exactly soluble [2], but due to the nonlinear nature of the TDGL equation, no exact solutions are available for a general number of dimensions, so we must rely on approximate theories. There are several approximate theories which describe the coarsening dynamics of this system. However, they all rely on a similar approach in which the order parameter ϕ is replaced by a smoothly varying auxiliary field m, which has the same sign as ϕ, and whose zeros define the domain walls. The equation of motion (TDGL) is then recast in terms of this auxiliary field, and various approximations are made in order to make the resulting equation soluble. In the Ohta-Jasnow-Kawasaki (OJK) [3] theory, this is achieved by replacing the nonlinear terms by their spherical average, thus linearizing the equation for the auxiliary field .

Comparison of the OJK results with simulation data [4] have shown that this theory gives a very good description of the system. However, the approximation is uncontrolled, and it is unclear why this approach gives such accurate predictions. Furthermore, Blundell, Bray, and Sattler [5] pointed out that by plotting $(1-C_\phi)$ against $1/C_{\phi^2}$, where

$$C_\phi=\langle \phi(\mathbf{x}+\mathbf{r},t)\phi(\mathbf{x},t)\rangle, \tag{1}$$

$$C_{\phi^2}=\frac{\langle[1-\phi^2(\mathbf{x}+\mathbf{r},t)][1-\phi^2(\mathbf{x},t)]\rangle}{\langle 1-\phi^2(\mathbf{x}+\mathbf{r},t)\rangle\langle 1-\phi^2(\mathbf{x},t)\rangle}, \tag{2}$$

thereby removing any adjustable parameters, the OJK theory does not follow the simulation data as closely as previous comparisons, using only C_ϕ, suggested. An obvious way to examine this approximation is to treat the neglected nonlinear terms as a perturbation to the linearized equation of motion, and this is exactly the approach taken in this paper.

This calculation has two main advantages: first, in OJK theory the initial conditions are conventionally taken to be Gaussian; since the evolution equation is linear, this will ensure that the distribution for the auxiliary field at later times is also Gaussian. This feature of OJK theory has been questioned by Yeung, Oono, and Shinozaki [6]. Their simulations, which explicitly calculate the auxiliary field distribution, gave results which are not exactly Gaussian, particularly at small values of the auxiliary field. We note that recently, in a series of papers, Mazenko [7,8], and Wickham and Mazenko [9] have presented an approximate theory which goes beyond the Gaussian distribution.

Second, it was proposed by both Bray and Humayun [10] and Liu and Mazenko [11] that the OJK approximation becomes exact as the number of dimensions approaches infinity. Evaluating the dimensional dependence of the first order correction term enables this hypothesis to be tested.

The main result of this paper is that the first order correction to the correlation function is $O(1/d^2)$, lending weight to the assertion that the OJK theory becomes exact in an infinite number of dimensions. The exact form of the leading order correction to the two-time correlation function, $C_1(\mathbf{r},t_1,t_2)$, is calculated in the large-d limit; this is found to obey both Porod's Law [12] and the Tomita sum rule [13], with the singular contribution in the correction term modifying the amplitude of the Porod tail, and in the limit $t_1 \gg t_2$, it is found that the correction term is of exactly the same form as the zero order result. The complexity of the calculation, however, has so far prevented the evaluation of higher order terms.

In Sec. II we present an outline of the OJK calculation, which provides the starting point for the perturbation calculation. Then, in Sec. III, we proceed with a detailed description of the perturbation calculation. We conclude with a summary and discussion of the results.

II. OJK THEORY

Consider a system described by a nonconserved scalar order parameter. The evolution of this system following a rapid quench from a high-temperature homogeneous phase to a regime where there are two equilibrium phases is described by the TDGL equation

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}=-\frac{\delta F[\phi]}{\delta\phi}, \tag{3}$$

where $F[\phi]$ is a Ginzburg-Landau free energy functional, given by

$$F[\phi]=\int d^d\mathbf{x}\,[\tfrac{1}{2}|\nabla\phi|^2+V(\phi)]. \tag{4}$$

$V(\phi)$ is a potential whose minima define the equilibrium values of the order parameter; the conventional choice in the case of a scalar order parameter is $V(\phi)=(1-\phi^2)^2/4$. Note that there is no noise term in this equation, hence the results are only valid for quenches to temperatures where the effect of thermal fluctuations is negligible.

Following the work of Ohta, Jasnow and Kawasaki [3], the scalar field $\phi(\mathbf{r},t)$ is replaced by a smoothly varying auxiliary field $m(\mathbf{r},t)$, where $\phi(\mathbf{r},t)$ and $m(\mathbf{r},t)$ have the same sign. The zeros of the field m then define the positions of the domain walls, and the normal $\hat{\mathbf{n}}$ to a domain wall in the direction of increasing ϕ is given by

$$\hat{\mathbf{n}}=\frac{\nabla m}{|\nabla m|}. \tag{5}$$

Inserting this into the Allen-Cahn equation for the velocity of an interface [14], $v=-K=-\nabla\cdot\hat{\mathbf{n}}$, where K is the curvature of the interface, we obtain

$$v=\frac{1}{|\nabla m|}\left(-\nabla^2 m+n_in_j\frac{\partial^2 m}{\partial x_i\partial x_j}\right), \tag{6}$$

where v is the speed of the interface in the direction of $\hat{\mathbf{n}}$. We now seek to obtain an evolution equation for $m(\mathbf{r},t)$ by linking the domain-wall velocity to the time dependence of the auxiliary field. The total time derivative of the auxiliary field in a frame of reference moving at a velocity $\mathbf{v}$ is given by

$$\frac{dm}{dt}=\frac{\partial m}{\partial t}+\mathbf{v}\cdot\nabla m. \tag{7}$$

If this frame is moving with the interface velocity, then $\mathbf{v}=v\hat{\mathbf{n}}$ and ∇m are parallel, and the total derivative of m vanishes, implying that

$$\frac{\partial m(\mathbf{r},t)}{\partial t}=-v|\nabla m|. \tag{8}$$

We now substitute the expression for the interface velocity [from Eq. (6)] into Eq. (8) to obtain

$$\frac{\partial m}{\partial t}=\nabla^2 m-n_in_j\frac{\partial^2 m}{\partial x_i\partial x_j}. \tag{9}$$

This is the time evolution equation for the auxiliary field obtained by OJK.

To make analytic progress, we now linearize this partial differential equation by replacing n_in_j by its spherical average δ_{ij}/d. In this approximation, the evolution of the field is governed by a simple diffusion equation with diffusion constant $D=1-1/d$. To calculate the correlation function we need to express the original order parameter ϕ in terms of the auxiliary field. In the thin wall limit this is given by $\phi=\mathrm{sgn}(m)$, therefore,

$$C(\mathbf{r},t_1,t_2)=\langle \mathrm{sgn}[m(\mathbf{x}+\mathbf{r},t_1)]\mathrm{sgn}[m(\mathbf{x},t_2)]\rangle. \tag{10}$$

It is convenient to choose Gaussian initial conditions for the auxiliary field, with zero mean, and correlator

$$\langle m(\mathbf{x}+\mathbf{r},0)m(\mathbf{x},0)\rangle=\Delta\,\delta(\mathbf{r}). \tag{11}$$

Since the evolution equation is linear, the auxiliary field will then have a Gaussian distribution at all times. Therefore, to calculate the correlation function we simply need to evaluate the joint probability distribution for $m(\mathbf{x}+\mathbf{r},t_1)\equiv m(1)$ and $m(\mathbf{x},t_2)\equiv m(2)$. This is given by

$$P[m(1),m(2)]=\frac{1}{2\pi[S(1)S(2)(1-\gamma^2)]^{1/2}}\times\exp\left[-\frac{1}{2(1-\gamma^2)}\left(\frac{m(1)^2}{S(1)}+\frac{m(2)^2}{S(2)}-2\gamma\frac{m(1)m(2)}{[S(1)S(2)]^{1/2}}\right)\right], \tag{12}$$

where γ is the normalized correlator:

$$\gamma=\frac{\langle m(1)m(2)\rangle}{\langle m(1)^2\rangle^{1/2}\langle m(2)^2\rangle^{1/2}} \tag{13}$$

$$=\left(\frac{4t_1t_2}{(t_1+t_2)^2}\right)^{d/4}\exp\left[-\frac{\mathbf{r}^2}{4D(t_1+t_2)}\right], \tag{14}$$

and $S(1)=\langle m(1)^2\rangle$ and $S(2)=\langle m(2)^2\rangle$ [1]. Completion of the average over the field m yields the final OJK result for the correlation function, and is given by

$$C(\mathbf{r},t_1,t_2)=\frac{2}{\pi}\sin^{-1}\gamma. \tag{15}$$

We note that this result only has a trivial dependence on the dimension d, through the diffusion constant $D=1-1/d$.

III. PERTURBATION THEORY

The starting point for this calculation is the OJK equation for the evolution of the auxiliary field $m(\mathbf{x},t)$ (derived in Sec. II),

$$\frac{\partial m}{\partial t}=\nabla^2 m-n_in_j\frac{\partial^2 m}{\partial x_i\partial x_j}, \tag{16}$$

where $\mathbf{n}=\nabla m/|\nabla m|$. As noted in Sec. II, this equation is highly nonlinear, and to make analytic progress OJK approximated the nonlinear term n_in_j by its spherical average, δ_{ij}/d, reducing Eq. (16) to the diffusion equation with diffusion constant $D=1-1/d$.

In this calculation, the terms dropped in this approximation are treated as a perturbation to the diffusion equation, and the first order correction to the correlation function is calculated. Therefore the equation we wish to solve is

$$\frac{\partial m}{\partial t}=D\nabla^2 m-\lambda\left(n_in_j-\frac{\delta_{ij}}{d}\right)\frac{\partial^2 m}{\partial x_i\partial x_j}, \tag{17}$$

where λ is a small perturbation parameter.

The solution of Eq. (17) can be expressed as a power series in λ, $m(\mathbf{r},t)=m_0+\lambda m_1+O(\lambda^2)$; substituting this back into Eq. (17) and equating orders of λ gives two coupled partial differential equations for m_0 and m_1, the solutions to which are given by

$$m_0(\mathbf{r},t)=\int d^d\mu G(\mathbf{r}-\boldsymbol{\mu},t)m_0(\boldsymbol{\mu},0), \tag{18}$$

$$m_1(\mathbf{r},t)=-\int_0^t d\tau\int d^d\mu G(\mathbf{r}-\boldsymbol{\mu},t-\tau)\left(\frac{\partial_i m_0(\boldsymbol{\mu},\tau)\partial_j m_0(\boldsymbol{\mu},\tau)}{|\boldsymbol{\nabla} m_0(\boldsymbol{\mu},\tau)|^2}-\frac{\delta_{ij}}{d}\right)\frac{\partial^2 m_0(\boldsymbol{\mu},\tau)}{\partial\mu_i\partial\mu_j}, \tag{19}$$

where $G(\mu-\mathbf{r},\tau-t)$ is the Green's function for the diffusion equation and is given by

$$G(\mathbf{r},t)=\frac{1}{(4\pi Dt)^{d/2}}\exp\left[-\frac{\mathbf{r}^2}{4Dt}\right], \tag{20}$$

and $\partial_i m_0(\boldsymbol{\mu},\tau)\equiv[\partial m_0(\boldsymbol{\mu},\tau)]/\partial\mu_i$.

In the thin wall limit the order parameter is related to the auxiliary field by the equation $\phi=\mathrm{sgn}(m)$. Hence, by using the integral representation of $\mathrm{sgn}(m)$ and expanding the expression for the correlation function in λ, we find that

$$C(\mathbf{r},t_1,t_2)=\langle\phi(\mathbf{x}+\mathbf{r},t_1)\phi(\mathbf{x},t_2)\rangle=C_0(\mathbf{r},t_1,t_2)+\lambda C_1(\mathbf{r},t_1,t_2)+O(\lambda^2), \tag{21}$$

where

$$C_0(\mathbf{r},t_1,t_2)=\langle\mathrm{sgn}[m_0(\mathbf{x}+\mathbf{r},t_1)]\mathrm{sgn}[m_0(\mathbf{x},t_2)]\rangle, \tag{22}$$

$$C_1(\mathbf{r},t_1,t_2)=2\{\langle\mathrm{sgn}[m_0(\mathbf{x}+\mathbf{r},t_1)]\delta[m_0(\mathbf{x},t_2)]m_1(\mathbf{x},t_2)\rangle+\langle\mathrm{sgn}[m_0(\mathbf{x},t_2)]\delta[m_0(\mathbf{x}+\mathbf{r},t_1)]m_1(\mathbf{x}+\mathbf{r},t_1)\rangle\}. \tag{23}$$

If we define $\tilde{C}_1(\mathbf{r},t_1,t_2)$ by

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=2\langle\mathrm{sgn}[m_0(\mathbf{x}+\mathbf{r},t_1)]\delta[m_0(\mathbf{x},t_2)]m_1(\mathbf{x},t_2)\rangle, \tag{24}$$

then the first order correction to the correlation function is given by

$$C_1(\mathbf{r},t_1,t_2)=\tilde{C}_1(\mathbf{r},t_1,t_2)+\tilde{C}_1(-\mathbf{r},t_2,t_1). \tag{25}$$

Since the two terms on the right-hand side of the expression for $C_1(\mathbf{r},t_1,t_2)$ [Eq. (25)] differ only in that $t_1\to t_2$ and $\mathbf{r}\to-\mathbf{r}$, we will only deal with one of these averaged terms, $\tilde{C}_1(\mathbf{r},t_1,t_2)$, evaluating the complete expression at the end of the calculation.

Substituting for m_1 from Eq. (19) into Eq. (24), we obtain the following expression for $\tilde{C}_1(\mathbf{r},t_1,t_2)$,

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=-2\int_0^{t_2}d\tau\int d^d\mu G(\mathbf{x}-\boldsymbol{\mu},t_2-\tau)\left\langle\mathrm{sgn}[m_0(\mathbf{x}+\mathbf{r},t_1)]\left(\frac{\partial_i m_0(\boldsymbol{\mu},\tau)\partial_j m_0(\boldsymbol{\mu},\tau)}{|\boldsymbol{\nabla} m_0(\boldsymbol{\mu},\tau)|^2}-\frac{\delta_{ij}}{d}\right)\delta[m_0(\mathbf{x},t_2)]\frac{\partial^2 m_0(\boldsymbol{\mu},\tau)}{\partial\mu_i\partial\mu_j}\right\rangle. \tag{26}$$

We now impose the conventional Gaussian initial conditions, with a zero mean, and a correlator given by

$$\langle m(\mathbf{x}+\mathbf{r},0)m(\mathbf{x},0)\rangle=\Delta\,\delta(\mathbf{r}). \tag{27}$$

The average on the right-hand side of Eq. (26) can be evaluated if we extract the differential operator $\partial^2/\partial\mu_i\partial\mu_i$ from the average by defining a new spatial variable $\boldsymbol{\nu}$. Equation (26) can then be written as

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=-2\int_0^{t_2}d\tau\int d^d\mu G(\mathbf{x}-\boldsymbol{\mu},t_2-\tau)\left.\frac{\partial^2 I_{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\nu}=\boldsymbol{\mu}}, \tag{28}$$

where

$$I_{ij}=\left\langle\left(\frac{\partial_i m_0(\boldsymbol{\mu},\tau)\partial_j m_0(\boldsymbol{\mu},\tau)}{|\boldsymbol{\nabla} m_0(\boldsymbol{\mu},\tau)|^2}-\frac{\delta_{ij}}{d}\right)\times\mathrm{sgn}[m_0(\mathbf{x}+\mathbf{r},t_1)]m_0(\boldsymbol{\nu},\tau)\delta[m_0(\mathbf{x},t_2)]\right\rangle. \tag{29}$$

The ensemble average in Eq. (29) can now be completed by using the joint probability distribution connecting the variables $m_0(\mathbf{x}+\mathbf{r},t_1)$, $m_0(\mathbf{x},t_2)$, $m_0(\boldsymbol{\nu},\tau)$, and $[\partial m_0(\boldsymbol{\mu},\tau)]/\partial\mu_i$.

For simplicity, at this point we introduce a contracted notation:

$$\frac{\partial m_0(\boldsymbol{\mu},\tau)}{\partial \mu_i}\to m'_i\,, \quad m_0(\boldsymbol{\nu},\tau)\to m(2), \tag{30}$$

$$m_0(\mathbf{x}+\mathbf{r},t_1)\to m(1), \quad m_0(\mathbf{x},t_2)\to m(3), \tag{31}$$

and define the vector $\tilde{\mathbf{m}}$ by

$$\tilde{\mathbf{m}}=[m'_1,m'_2,m'_3,\ldots,m'_d,m(1),m(2),m(3)]. \tag{32}$$

The joint probability distribution for the components of this vector is then given by

$$P(\tilde{\mathbf{m}})=\frac{1}{(2\pi)^{(d+3)/2}(\det A^{-1})^{1/2}}\exp\left(-\frac{1}{2}\tilde{m}_i A_{ij}\tilde{m}_j\right), \tag{33}$$

where $A^{-1}_{ij}=\langle \tilde{m}_i\tilde{m}_j\rangle$. $P(\tilde{\mathbf{m}})$ is evaluated in Appendix A, and is given by

$$P(\tilde{\mathbf{m}})=\frac{1}{(2\pi)^{(d+3)/2}(\det A^{-1})^{1/2}}\exp\left[-\frac{F(\tilde{\mathbf{m}})}{2\det A^{-1}}\right], \tag{34}$$

where $F(\tilde{\mathbf{m}})$ is given by Eq. (A32).

Equation (29) may therefore be written in the form

$$I_{ij}=\int d^d\mathbf{m}'\int_{-\infty}^{\infty}dm(1)\int_{-\infty}^{\infty}dm(2)\int_{-\infty}^{\infty}dm(3)P(\tilde{\mathbf{m}})\times\left(\frac{m'_i m'_j}{|\mathbf{m}'|^2}-\frac{\delta_{ij}}{d}\right)\mathrm{sgn}[m(1)]m(2)\,\delta[m(3)]. \tag{35}$$

The function $\tilde{C}_1(\mathbf{r},t_1,t_2)$ defined by Eq. (28) is evaluated in three main steps. First, the expression for I_{ij} [Eq. (35)] is evaluated. This result is then differentiated to obtain $\partial^2 I_{ij}/\partial\nu_i\partial\nu_j|_{\boldsymbol{\nu}=\boldsymbol{\mu}}$, and, finally, this expression is substituted back into Eq. (28) and the remaining integrals are completed.

To evaluate Eq. (35), we must complete the integrals over $\tilde{\mathbf{m}}$. The integral over $m(3)$ is trivial; on setting $m(3)$ to zero in Eq. (35), we see that the integral over $m(2)$ can be reduced to a Gaussian by completing the square in the exponent of the probability density function, $-F(\tilde{\mathbf{m}})/2\det A^{-1}$. For simplicity we deal with the expression for $F(\tilde{\mathbf{m}})$ [Eq. (A32)] first. Completing the square using the substitution $m'(2)=m(2)+[sm(1)+\boldsymbol{\eta}\cdot\mathbf{m}']/q$ in $m(2)$ gives

$$F(\tilde{m})=m'_k\left(\theta_{kl}-\frac{\eta_k\eta_l}{q}\right)m'_l+\frac{(pq-s^2)}{q}m(1)^2-2\frac{(s\boldsymbol{\eta}-q\boldsymbol{\xi})}{q}\cdot\mathbf{m}'m(1)+qm'(2)^2. \tag{36}$$

This may be simplified further by substituting for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in the factor $s\boldsymbol{\eta}-q\boldsymbol{\xi}$, from Eqs. (A27) and (A28), respectively, to give

$$s\boldsymbol{\eta}-q\boldsymbol{\xi}=(pq-s^2)\mathbf{a}-(us-qt)\mathbf{c}=(pq-s^2)\mathbf{h}, \tag{37}$$

where this equation is used to define the vector $\mathbf{h}$. Using Eqs. (A42) and (A44), $\mathbf{h}$ may then be written as

$$\mathbf{h}=\mathbf{a}-\left(\frac{us-qt}{pq-s^2}\right)\mathbf{c}=\mathbf{a}-\left(\frac{zy-\mathbf{a}\cdot\mathbf{c}}{z\lambda_3-\mathbf{c}^2}\right)\mathbf{c}. \tag{38}$$

Substituting Eq. (37) back into Eq. (36), $F(\tilde{\mathbf{m}})$ reduces to

$$F(\tilde{m})=m'_k\left(\theta_{kl}-\frac{\eta_k\eta_l}{q}\right)m'_l+\frac{(pq-s^2)}{q}[m(1)^2-2\mathbf{h}\cdot\mathbf{m}_\prime m(1)]+qm'(2)^2. \tag{39}$$

On substituting Eqs. (34) and (39) back into Eq. (35) we find that the $m(2)$ integral is now in the form of a Gaussian, on completion of which, I_{ij} simplifies to

$$I_{ij}=\frac{-(2\pi)^{-(d+2)/2}}{q^{3/2}}\int d^d\mathbf{m}'\left(\frac{m'_i m'_j}{|\mathbf{m}'|^2}-\frac{\delta_{ij}}{d}\right)\exp\left[-\frac{1}{2\det A^{-1}}m'_k\left(\theta_{kl}-\frac{\eta_k\eta_l}{q}\right)m'_l\right]\times\int_{-\infty}^{\infty}dm(1)\,\mathrm{sgn}[m(1)](sm(1)+\boldsymbol{\eta}\cdot\mathbf{m}')\exp\left[-\frac{(pq-s^2)}{2q\det A^{-1}}[m(1)^2-2\mathbf{h}\cdot\mathbf{m}'m(1)]\right]. \tag{40}$$

We now manipulate this expression into a form in which the $\mathbf{m}'$ integral can be completed. First we need to remove the $|\mathbf{m}'|^2$ term from the denominator; this can be achieved by rewriting the denominator as an integral over an exponential,

$$\frac{1}{|\mathbf{m}'|^2}=\int_0^{\infty}d\tilde{v}\ \exp[-\tilde{v}|\mathbf{m}'|^2]. \tag{41}$$

The second step is to manipulate the $m(1)$ dependence so that the exponent in the complete expression for I_{ij} can be written in the form $\mathbf{m}'^T\Omega\mathbf{m}'$, where Ω is a $d\times d$ matrix to be determined later.

Due to the presence of the $\mathrm{sgn}[m(1)]$ function, the positive and negative ranges of the integral must be treated separately; in each case we complete the square in the exponent using the substitution $\tilde{u}=[m(1)-\mathbf{h}\cdot\mathbf{m}']/\mathbf{h}\cdot\mathbf{m}'$. The antisymmetric contribution to the $m(1)$ integral in Eq. (40) may be evaluated directly, and the symmetric contribution reduces to a term proportional to an error function. This integral, therefore, reduces to

$$\int dm(1)\mathrm{sgn}[m(1)][sm(1)+\boldsymbol{\eta}\cdot\mathbf{m}')\exp\left[-\frac{(pq-s^2)}{2q\det A^{-1}}[m(1)^2-2\mathbf{h}\cdot\mathbf{m}'m(1)]\right]$$
$$=\alpha+\beta_{kl}m'_k m'_l\int_0^1 d\tilde{u}\ \exp\left[-\frac{(pq-s^2)}{2q\det A^{-1}}(\mathbf{h}\cdot\mathbf{m}')^2(\tilde{u}^2-1)\right], \tag{42}$$

where

$$\alpha=\frac{2qs\ \det A^{-1}}{(pq-s^2)}$$

and

$$\beta_{kl}=2h_k(\eta_l+sh_l). \tag{43}$$

Substituting Eqs. (41) and (42) back into the expression for I_{ij} [Eq. (40)], we obtain

$$I_{ij}=\frac{1}{(2\pi)^{(d+2)/2}q^{3/2}}\int d^d\mathbf{m}'\ \exp\left[-\frac{1}{2\det A^{-1}}m'_e\left(\theta_{ef}-\frac{\eta_e\eta_f}{q}\right)m'_f\right]\left(\frac{\delta_{ij}}{d}-m'_i m'_j\int_0^\infty d\tilde{v}\ \exp[-\tilde{v}|\mathbf{m}'|^2]\right)$$
$$\times\left(\alpha+\beta_{kl}m'_k m'_l\int_0^1 d\tilde{u}\ \exp\left[-\frac{(pq-s^2)}{2q\ \det A^{-1}}(\mathbf{h}\cdot\mathbf{m}')^2(\tilde{u}^2-1)\right]\right). \tag{44}$$

We are now in a position to define the general matrix $\Omega(\tilde{u},\tilde{v})$ referred to earlier:

$$\Omega_{ij}(\tilde{u},\tilde{v})=\frac{1}{\det A^{-1}}\left(\theta_{ij}-\frac{\eta_i\eta_j}{q}+\frac{(pq-s^2)}{q}h_ih_j(\tilde{u}^2-1)\right)+2\tilde{v}\,\delta_{ij}\,. \tag{45}$$

By expanding the right-hand side of Eq. (44), we see that the exponent of each contribution is given by the general $d\times d$ matrix $\Omega(\tilde{u},\tilde{v})$, evaluated at different values of $\tilde{u}$ and $\tilde{v}$. Equation (44) can therefore be written as the sum of four standard integrals,

$$I_{ij}=\sum_{n=1}^{4}I_n^{ij}, \tag{46}$$

where

$$I_1^{ij}=\frac{\alpha\delta_{ij}}{d(2\pi)^{(d+2)/2}q^{3/2}}\int d^d\mathbf{m}'\exp\left[-\frac{1}{2}\mathbf{m}'^T\Omega(1,0)\mathbf{m}'\right]=\frac{\alpha\delta_{ij}}{2\pi dq^{3/2}[\det\Omega(1,0)]^{1/2}}, \tag{47}$$

$$I_2^{ij}=-\frac{\alpha}{(2\pi)^{(d+2)/2}q^{3/2}}\int_0^\infty d\tilde{v}\int d^d\mathbf{m}'m'_i m'_j\exp'=-\frac{\alpha}{2\pi q^{3/2}}\int_0^\infty d\tilde{v}\frac{\Omega_{ij}^{-1}(1,\tilde{v})}{[\det\Omega(1,\tilde{v})]^{1/2}}, \tag{48}$$

$$I_3^{ij}=\frac{\beta_{kl}\delta_{ij}}{d(2\pi)^{(d+2)/2}q^{3/2}}\int_0^1 d\tilde{u}\int d^d\mathbf{m}'m'_k m'_l\exp\left[-\frac{1}{2}\mathbf{m}'^T\Omega(\tilde{u},0)\mathbf{m}'\right]=\frac{\delta_{ij}\beta_{kl}}{2\pi dq^{3/2}}\int_0^1 d\tilde{u}\frac{\Omega_{kl}^{-1}(\tilde{u},0)}{[\det\Omega(\tilde{u},0)]^{1/2}}, \tag{49}$$

$$I_4^{ij}=-\frac{\beta_{kl}}{(2\pi)^{(d+2)/2}q^{3/2}}\int_0^\infty d\tilde{v}\int_0^1 d\tilde{u}\int d^d\mathbf{m}'m'_i m'_j m'_k m'_l\exp\left[-\frac{1}{2}\mathbf{m}'^T\Omega(\tilde{u},\tilde{v})\mathbf{m}'\right]$$
$$=-\frac{\beta_{kl}}{2\pi q^{3/2}}\int_0^\infty d\tilde{v}\int_0^1 d\tilde{u}\frac{1}{[\det\Omega(\tilde{u},\tilde{v})]^{1/2}}(\Omega_{ij}^{-1}(\tilde{u},\tilde{v})\Omega_{kl}^{-1}(\tilde{u},\tilde{v})+\Omega_{ik}^{-1}(\tilde{u},\tilde{v})\Omega_{jl}^{-1}(\tilde{u},\tilde{v})+\Omega_{il}^{-1}(\tilde{u},\tilde{v})\Omega_{jk}^{-1}(\tilde{u},\tilde{v})). \tag{50}$$

Details of the calculation of the inverse and determinant of $\Omega(\tilde{u},\tilde{v})$ are supplied in Appendix B, and are given by equations (B27)–(B14).

At this point we can calculate $\partial^2 I_{ij}/\partial\nu_i\partial\nu_j|_{\boldsymbol{\nu}=\boldsymbol{\mu}}$, which once completed will be substituted into Eq. (28). First, we notice that the only variables which depend on $\boldsymbol{\nu}$ in the expressions for the I_n^{ij} are α and β_{kl} [defined in Eq. (43)]. Our first step, therefore, is to simplify these quantities and then calculate their derivatives.

Inserting Eq. (A42) into the expression for α, and differentiating, we find

$$\left.\frac{\partial^2\alpha}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \frac{2q}{(z\lambda_3-\mathbf{c}^2)}\left(\frac{z}{\lambda}\right)^{d+1}\left.\frac{\partial^2 s}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}, \tag{51}$$

(From this point on we will use the notation $\partial^2\alpha/\partial\nu_i\partial\nu_j|_{\boldsymbol{\mu}}$ to indicate that we are evaluating the differential at $\boldsymbol{\nu}=\boldsymbol{\mu}$.) To simplify β_{kl}, we substitute for $\mathbf{h}$ and $\boldsymbol{\eta}$ from Eqs. (38) and (A28) respectively; using Eq. (A40), we find that

$$\beta_{kl} = \frac{2q}{z\lambda_3-\mathbf{c}^2}\left[a_k - \left(\frac{zy-\mathbf{a}\cdot\mathbf{c}}{z\lambda_3-\mathbf{c}^2}\right)c_k\right] \times[(zv-\mathbf{b}\cdot\mathbf{c})c_l-(z\lambda_3-\mathbf{c}^2)b_l]. \tag{52}$$

We now note that, from the definition of $\mathbf{b}$ [Eq. (A8)], $\partial^2\mathbf{b}/\partial\nu_i\partial\nu_j|_{\boldsymbol{\mu}}=0$, so the derivative of β_{kl} is given by

$$\left.\frac{\partial^2\beta_{kl}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \frac{2qz}{(z\lambda_3-\mathbf{c}^2)}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} h_k c_l. \tag{53}$$

Differentiating Eqs. (47)–(50) and using Eqs. (B14), (51), and (53), we obtain the following expressions for the $\partial^2 I_n^{ij}/\partial\nu_i\partial\nu_j|_{\boldsymbol{\mu}}$:

$$\left.\frac{\partial^2 I_1^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \frac{\delta_{ij}}{d\pi(z\lambda_3-\mathbf{c}^2)[\tilde{q}\Delta(1,0)]^{1/2}}\left(\frac{z}{\lambda}\right)^{d+2}\left.\frac{\partial^2 s}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}, \tag{54}$$

$$\left.\frac{\partial^2 I_2^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \frac{-1}{\pi(z\lambda_3-\mathbf{c}^2)}\left(\frac{z}{\lambda}\right)^{d+2}\left.\frac{{}^2 s}{\partial\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} \times\int_0^\infty \frac{d\tilde{v}\,\Omega_{ij}^{-1}(1,\tilde{v})}{\Lambda^{(d-2)/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}}, \tag{55}$$

$$\left.\frac{\partial^2 I_3^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \frac{z^2\delta_{ij}h_k c_l}{d\pi\lambda(z\lambda_3-\mathbf{c}^2)}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\int_0^1 d\tilde{u}\frac{\Omega_{kl}^{-1}(\tilde{u},0)}{[\tilde{q}\Delta(\tilde{u},0)]^{1/2}}, \tag{56}$$

$$\left.\frac{\partial^2 I_4^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} \frac{-z^2 h_k c_l}{\pi\lambda(z\lambda_3-\mathbf{c}^2)}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\int_0^\infty d\tilde{v}\int_0^1 d\tilde{u}\frac{1}{\Lambda^{(d-2)/2}[\tilde{q}\Delta(\tilde{u},\tilde{v})]^{1/2}}[\Omega_{ij}^{-1}(\tilde{u},\tilde{v})\Omega_{kl}^{-1}(\tilde{u},\tilde{v})+\Omega_{ik}^{-1}(\tilde{u},\tilde{v})\Omega_{jl}^{-1}(\tilde{u},\tilde{v}) +\Omega_{il}^{-1}(\tilde{u},\tilde{v})\Omega_{jk}^{-1}(\tilde{u},\tilde{v})]. \tag{57}$$

At this point we shall pause to give an overview of the simplification of Eqs. (54)–(57). We first consider Eqs. (56) and (57); on completing the contraction over the k and l indices, we obtain expressions in which the $\tilde{u}$ integration may be completed exactly. We then complete the contractions over the i and j indices in the expression

$$\left.\frac{\partial^2 I_{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \sum_{n=1}^{4}\left.\frac{\partial^2 I_n^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}, \tag{58}$$

and substitute this back into the expression for $\tilde{C}_1(\mathbf{r},t_1,t_2)$ [Eq. (28)].

Before we complete the contraction over the k and l indices in Eqs. (56) and (57), we will derive some results which will be used later. Using the definitions of $\mathbf{h}$, Δ, $\mathbf{k}$, and $\tilde{q}$ [Eqs. (38), (B13), (B26), and (B7), respectively], we find

$$\mathbf{k}(\tilde{v})\cdot\mathbf{h} = \frac{\tilde{q}}{\tilde{u}^2}\left[\Delta(\tilde{u},\tilde{v}) - \Lambda\left(\frac{\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2}{z\lambda_3-\mathbf{c}^2}\right)\right]. \tag{59}$$

Using Eq. (59) together with the definitions of $\Omega^{-1}(\tilde{u},\tilde{v})$, $\mathbf{h}$ and $\mathbf{k}$ [Eqs. (B27), (38) and (B26), respectively] we also obtain:

$$\Omega_{ij}^{-1}(\tilde{u},\tilde{v})h_j = \frac{\lambda k_i(\tilde{v})}{z(z\lambda_3-\mathbf{c}^2)\Delta(\tilde{u},\tilde{v})}, \tag{60}$$

$$\Omega_{ij}^{-1}(\tilde{u},\tilde{v})c_j = \frac{\lambda}{z[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}\left((z\lambda_3-\mathbf{c}^2)c_j - \frac{\mathbf{k}(\tilde{v})\cdot\mathbf{c}\tilde{u}^2}{\Lambda\tilde{q}\Delta(\tilde{u},\tilde{v})}k_j(\tilde{v})\right). \tag{61}$$

Substituting Eqs. (60) and (61) into Eqs. (56) and (57), we can therefore complete the contractions over the k and l indices to obtain:

$$\left.\frac{\partial^2 I_3^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = \frac{z\mathbf{k}(0)\cdot\mathbf{c}\delta_{ij}}{d\pi\tilde{q}^{1/2}(z\lambda_3-\mathbf{c}^2)^2}\left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}\int_0^1 \frac{d\tilde{u}}{\Delta(\tilde{u},0)^{3/2}}, \tag{62}$$

$$\left.\frac{\partial^2 I_4^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\frac{z}{\pi\tilde{q}^{1/2}(z\lambda_3-\mathbf{c}^2)^2}\left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}\int_0^\infty\int_0^1 \frac{d\tilde{u}d\tilde{v}}{\Lambda^{(d-2)/2}\Delta(\tilde{u},\tilde{v})^{3/2}}\left[\frac{\lambda\mathbf{k}(\tilde{v})\cdot\mathbf{c}}{\Lambda z}\left(\delta_{ij}-\frac{c_i c_j}{\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2}\right.\right.$$
$$\left.\left.-\frac{3k_i(\tilde{v})k_j(\tilde{v})\tilde{u}^2}{\tilde{q}\Delta(\tilde{u},\tilde{v})[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}\right)+\frac{\lambda(z\lambda_3-\mathbf{c}^2)[c_i k_j(\tilde{v})+c_j k_i(\tilde{v})]}{z[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}\right]. \tag{63}$$

Equations (62) and (63) are now in a form in which we can complete the $\tilde{u}$ integration. We see that the only integrals required are $\int_0^1 d\tilde{u}\Delta(\tilde{u},\tilde{v})^{-3/2}$ and $\int_0^1 d\tilde{u}\tilde{u}^{\,2}\Delta(\tilde{u},\tilde{v})^{-5/2}$; since $\Delta(\tilde{u},\tilde{v})$ [which is defined by Eq. (B13)] is of the form $\gamma+\delta\tilde{u}^2$, these integrals over $\tilde{u}$ may easily be completed to give

$$\int_0^1 d\tilde{u}\frac{1}{[\Delta(\tilde{u},\tilde{v})]^{3/2}} = \frac{(z\lambda_3-\mathbf{c}^2)}{\Lambda[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2][\Delta(1,\tilde{v})]^{1/2}}, \tag{64}$$

$$\int_0^1 d\tilde{u}\frac{\tilde{u}^{\,2}}{[\Delta(\tilde{u},\tilde{v})]^{5/2}} = \frac{(z\lambda_3-\mathbf{c}^2)}{3\Lambda[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2][\Delta(1,\tilde{v})]^{3/2}}, \tag{65}$$

where $\tilde{q}\Delta(1,\tilde{v})=\Lambda^2 z^2(\lambda_1\lambda_3-y^2)-\Lambda(\Lambda-1)z(\lambda_3\mathbf{a}^2-2y\mathbf{a}\cdot\mathbf{c}+\lambda_1\mathbf{c}^2)+(\Lambda-1)^2(\mathbf{a}^2\mathbf{c}^2-\mathbf{a}\cdot\mathbf{c}^2)$. We can therefore substitute Eqs. (64) and (65) into the right-hand side of Eqs. (62) and (63) to obtain

$$\left.\frac{\partial^2 I_3^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = \frac{\mathbf{k}(0)\cdot\mathbf{c}\delta_{ij}}{d\pi\lambda_3(z\lambda_3-\mathbf{c}^2)[\tilde{q}\Delta(1,0)]^{1/2}}\left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}, \tag{66}$$

$$\left.\frac{\partial^2 I_4^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\frac{z}{\pi(z\lambda_3-\mathbf{c}^2)}\left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}\int_0^\infty \frac{d\tilde{v}}{\Lambda^{d/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}}\left(\frac{\mathbf{k}(\tilde{v})\cdot\mathbf{c}\Omega_{ij}^{-1}(1,\tilde{v})}{[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}+\frac{\lambda(z\lambda_3-\mathbf{c}^2)[k_i(\tilde{v})c_j+c_i k_j(\tilde{v})]}{z[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]^2}\right). \tag{67}$$

We now substitute Eqs. (54), (55), (66), and (67) into Eq. (58), and complete the contraction over the i and j indices. The details of this calculation are contained within Appendix C. Following the contraction over the i and j indices, Eq. (58) reduces to

$$\left.\frac{\partial^2 I_{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\frac{\lambda}{\pi z}\int_0^\infty d\tilde{v}\sum_{n=1}^{6} T_n, \tag{68}$$

where the terms T_n are defined by Eqs. (C28)–(C33). Substituting Eqs. (20) and (68) into Eq. (28), where without loss of generality we can set $\mathbf{x}=\mathbf{0}$, we obtain the following expression for $\tilde{C}_1(\mathbf{r},t_1,t_2)$:

$$\tilde{C}_1(\mathbf{r},t_1,t_2) = \int_0^{t_2} d\tau\int_0^\infty d\tilde{v}\int d^d\mu\,\frac{2\lambda\sum_{n=1}^{6} T_n}{\pi z(4\pi D(t_2-\tau))^{d/2}}\times\exp\left(-\frac{\mu^2}{4D(t_2-\tau)}\right). \tag{69}$$

The spatial integral over μ may be completed by transforming to spherical polars and choosing a change of variables which allows the integral to be completed by steepest descents. In spherical polars the spatial integral may be written as

$$\int d^d\mu = \frac{2(\pi)^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty d\mu\;\mu^{d-1}\int_0^\pi d\psi(\sin\psi)^{d-2}. \tag{70}$$

We notice that each expression for T_n [Eqs. (C28)-(C33)] contains the factor $\Lambda^{-d/2}=(1+2\lambda\tilde{v}/z)^{-d/2}$; we therefore make the substitution $\tilde{w}=\tilde{v}\lambda d/z$, so that, in the large-$d$ limit, this factor reduces to an exponential in $\tilde{w}$. We also make the additional substitution $\tau=t_2(1-x/d)$, and rescale μ to $\hat{\mu}=(xt_2)^{-1/2}\mu$. The form of the τ substitution is chosen in the expectation that the contribution to the integral from later times will dominate the result. Equation (69) then reduces to

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=\frac{4t_2}{\pi^{3/2}d^2\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{d}{4D}\right)^{d/2}\times\sum_{n=1}^{6}\int_0^d dx\int_0^\infty d\tilde{w}\int_0^\infty\frac{d\hat{\mu}}{\hat{\mu}}\int_0^\pi\frac{d\psi}{(\sin\psi)^2}\times\exp[-dg(\hat{\mu},\psi)]T_n, \tag{71}$$

where $g(\hat{\mu},\psi)=(\hat{\mu}^2/4D)-\ln(\hat{\mu}\sin\psi)$.

Once the integral is in this form, the $\hat{\mu}$ and ψ integrals can be completed in the large-d limit by steepest descents. Applying this limit does not represent a serious limitation on the calculation, since this is the regime of most interest, as explained in Sec. I. In the limit $d\to\infty$, provided the large-d behavior of the terms T_i is controlled (as demonstrated in Appendix D 1), the value of the integral is dominated by the contribution from the neighborhood around the minima of the function $g(\hat{\mu},\psi)$.

Within the range of integration, the function $g(\hat{\mu},\psi)$ has a global minimum at $\hat{\mu}=\sqrt{2D}$ and $\psi=\pi/2$. We can expand the integrand to first order about this minimum, reducing the $\hat{\mu}$ and ψ integrations in Eq. (71) into Gaussians, and giving

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=\frac{4t_2\exp(-d/2)}{(2D)^{1/2}\pi^{3/2}d^2\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{d}{2}\right)^{d/2}\times\sum_{n=1}^{6}\int_0^d dx\int_0^\infty d\tilde{w}\,T_n\big|_{\mu=\sqrt{2D},\psi=\pi/2}\times\int_0^\infty d\hat{\mu}\exp\left(\frac{-d(\hat{\mu}-\sqrt{2D})^2}{2D}\right)\times\int_0^\pi d\psi\exp\left(\frac{-d(\psi-\pi/2)^2}{2}\right). \tag{72}$$

Since we are only interested in the first order correction to the correlation function, we use Stirling's formula to evaluate the leading order contribution from the γ function, which is given by

$$\Gamma\left(\frac{d-1}{2}\right)\sim\frac{2(2\pi)^{1/2}}{d}\left(\frac{d}{2}\right)^{d/2}\exp(-d/2). \tag{73}$$

On substituting Eq. (73) back into Eq. (72), and completing the Gaussian integrals, we find

$$\tilde{C}_1(\mathbf{r},t_1,t_2)=\frac{2t_2}{\pi d^2}\int_0^\infty\int_0^\infty dx\,d\tilde{w}\sum_{n=1}^{6}T_n\big|_{\mu=\sqrt{2D},\psi=\pi/2}. \tag{74}$$

Now all that remains is to calculate the leading order contribution from the integration over x and $\tilde{w}$ of the terms T_n, evaluated at $\mu=\sqrt{2D}$ and $\psi=\pi/2$. Each of the terms may be expanded as a power series in $1/d$, and in Appendix D 2 each term is evaluated to $O(1)$, giving

$$T_1=\frac{\gamma(t_2-t_1)\exp(-x/2)\tilde{w}\exp(-\tilde{w})}{8Dt_2(t_1+t_2)(1-\gamma^2)^{3/2}}\times\left[x\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)+\frac{2t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right], \tag{75}$$

$$T_2=\frac{\gamma(t_2-t_1)x\exp(-x/2)\tilde{w}\exp(-\tilde{w})}{4Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{76}$$

$$T_3=\frac{\gamma(t_2-t_1)x(x-2)\exp(-x/2)\exp(-\tilde{w})}{8Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{77}$$

$$T_4=\frac{-\gamma\mathbf{r}^2x\exp(-x/2)\exp(-\tilde{w})}{8D^2(t_1+t_2)^2(1-\gamma^2)^{1/2}}, \tag{78}$$

$$T_5=\frac{-\gamma\exp(-x/2)\exp(-\tilde{w})}{8Dt_2(1-\gamma^2)^{3/2}}\times\left[x\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)+\frac{2t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right]\times\left[\frac{t_2-t_1}{t_1+t_2}-\frac{t_2\mathbf{r}^2}{D(t_1+t_2)^2}+\frac{x}{2}\left(1-\frac{4t_2^2}{(t_1+t_2)^2}\right)\right], \tag{79}$$

$$T_6=0. \tag{80}$$

We now complete the x and $\tilde{w}$ integrations of the expressions given in Eqs. (75)–(80), finding

$$\int_0^\infty\int_0^\infty dx\,d\tilde{w}\,T_1=\frac{\gamma(t_2-t_1)}{2Dt_2(t_1+t_2)(1-\gamma^2)^{3/2}}\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}+\frac{t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right), \tag{81}$$

$$\int_0^\infty\int_0^\infty dx\,d\tilde{w}\,T_2=\frac{\gamma(t_2-t_1)}{Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{82}$$

$$\int_0^\infty\int_0^\infty dx\,d\tilde{w}\,T_3=\frac{\gamma(t_2-t_1)}{Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{83}$$

$$\int_0^\infty\int_0^\infty dx\, d\tilde{w}\; T_4 = \frac{-\gamma \mathbf{r}^2}{2D^2(t_1+t_2)^2(1-\gamma^2)^{1/2}}, \tag{84}$$

$$\int_0^\infty\int_0^\infty dx\, d\tilde{w}\; T_5 = \frac{-\gamma}{2Dt_2(1-\gamma^2)^{3/2}}\left[\left(1-\frac{4t_2^2}{(t_1+t_2)^2}\right)\left(2-\frac{8t_1t_2\gamma^2}{(t_1+t_2)^2}+\frac{t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right)\right.$$
$$\left.+\left(\frac{t_2-t_1}{t_1+t_2}-\frac{t_2\mathbf{r}^2}{D(t_1+t_2)^2}\right)\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}+\frac{t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right)\right], \tag{85}$$

$$\int_0^\infty\int_0^\infty dx\, d\tilde{w}\; T_6 = 0. \tag{86}$$

On substituting Eqs. (81)–(86) into Eq. (74), we obtain

$$\tilde{C}_1(\mathbf{r},t_1,t_2) = \frac{2\gamma}{D\pi d^2(1-\gamma^2)^{3/2}}\left[\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)\left(\frac{4t_2^2}{(t_1+t_2)^2}-1\right)+\frac{2(t_2-t_1)(1-\gamma^2)}{(t_1+t_2)}+\frac{2t_2^2(t_2-t_1)\gamma^2\mathbf{r}^2}{D(t_1+t_2)^4}+\frac{t_2^2\gamma^2\mathbf{r}^4}{2D^2(t_1+t_2)^4}\right]. \tag{87}$$

Finally, on substituting Eq. (87) back into Eq. (25), we obtain the complete expression for the first order correction to the correlation function:

$$C_1(\mathbf{r},t_1,t_2) = \frac{4\gamma}{D\pi d^2(1-\gamma^2)^{3/2}}\left[\left(\frac{t_1-t_2}{t_1+t_2}\right)^2\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}+\frac{\gamma^2\mathbf{r}^2}{D(t_1+t_2)}\right)+\frac{(t_1^2+t_2^2)\gamma^2\mathbf{r}^4}{4D^2(t_1+t_2)^4}\right]. \tag{88}$$

We now express our final result for the first order correction to the correlation function in the same form as the exact OJK result [Eq. (15)]. In this form the correlation function is given by

$$C(\mathbf{r},t_1,t_2) = \frac{2}{\pi}\sin^{-1}\{\gamma[1+\lambda F(\mathbf{r},t_1,t_2)]\}, \tag{89}$$

where

$$F(\mathbf{r},t_1,t_2) = \frac{2}{Dd^2(1-\gamma^2)}\left[\frac{(t_1-t_2)^2}{(t_1+t_2)^2}\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}+\frac{\gamma^2\mathbf{r}^2}{D(t_1+t_2)}\right)+\frac{(t_1^2+t_2^2)\gamma^2\mathbf{r}^4}{4D^2(t_1+t_2)^4}\right]. \tag{90}$$

We can clearly see from this expression that the first order correction to the correlation function is $O(1/d^2)$, which lends weight to the assertion that the OJK result becomes exact in an infinite-dimensional system.

Two special cases

We now evaluate this result in two special cases: at equal times, and when the times are widely separated, $t_1 \gg t_2$. At equal times, the first order correction term is given by

$$C_1(\mathbf{r},t) = \frac{1}{8D\pi d^2}\, g\left(\frac{r}{(Dt)^{1/2}}\right), \tag{91}$$

where

$$g(x) = \frac{x^4\exp(-3x^2/8)}{[1-\exp(-x^2/4)]^{3/2}}. \tag{92}$$

This correction term clearly exhibits the expected scaling, $L \sim t^{1/2}$; the scaling function $g(x)$ is shown in Fig. 1.

Figure 2 shows a comparison of the zero order OJK result [Eq. (15)] with the perturbed result for $d=2$,3 and 4 at equal times; the functions have all been scaled so that they have the same gradient at the origin. Although the result is only valid for large d, the figure clearly demonstrates that the perturbation will have the effect of lowering the exact OJK result. This is discussed further in Sec. IV.

If we now expand Eq. (91) about $r=0$, we see that, for small r,

$$C_1(\mathbf{r},t) \sim \frac{1}{D\pi d^2}\frac{r}{(Dt)^{1/2}}\left(1-\frac{3r^2}{16Dt}+\frac{7r^4}{512D^2t^2}\right.$$
$$\left.+O[r^6/(Dt)^3]\right), \tag{93}$$

and hence the result obeys Porod's Law [12] and the Tomita sum rule [13].

We now consider the case where the times are widely separated. If $t_1 \gg t_2$, then Eq. (88) reduces to

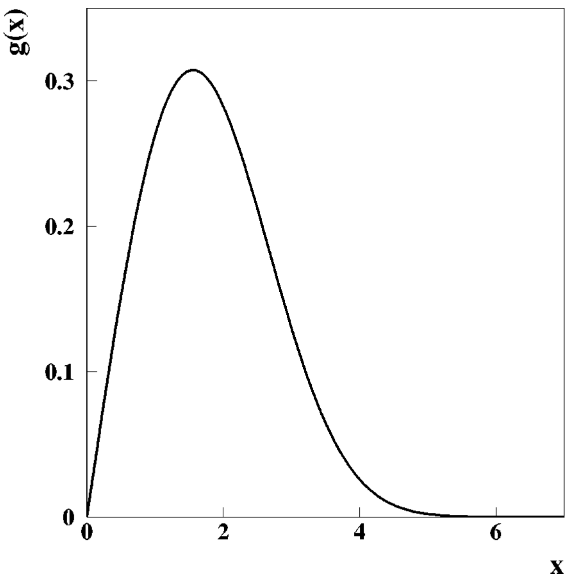

FIG. 1. Scaling function for the first order correction to the correlation function.

$$C_1(\mathbf{r},t_1,t_2)=\frac{4}{D\pi d^2}\left(\frac{4t_2}{t_1}\right)^{d/4}\exp\left(-\frac{\mathbf{r}^2}{4Dt_1}\right). \tag{94}$$

Comparing this to the conventional scaling form,

$$C_1(\mathbf{r},t_1,t_2)\sim\left(\frac{L_2}{L_1}\right)^{\bar{\lambda}}h\left(\frac{r}{L_1}\right),\quad t_1\gg t_2, \tag{95}$$

where L_1 and L_2 are the characteristic lengths for the system at times t_1 and t_2, respectively, we find that $\bar{\lambda}=d/2$, as expected. We also note that the leading order correction to the correlation function in this limit has exactly the same form as the zero order correlation function; from Eq. (15), we have

$$C_0(\mathbf{r},t_1,t_2)=\frac{2}{\pi}\left(\frac{4t_2}{t_1}\right)^{d/4}\exp\left(-\frac{\mathbf{r}^2}{4Dt_1}\right). \tag{96}$$

IV. DISCUSSION

In this paper I have described a perturbation theory applicable to OJK theory in a system with an infinite number of dimensions and no noise. The leading order correction term to the two-time correlation function has been calculated by treating the nonlinear terms in the OJK auxiliary-field evolution equation as a perturbation to the linearized equation. There are two separate motivations for this study. First, since in conventional OJK theory, the evolution equation is linear and Gaussian initial conditions are imposed, it follows that the distribution of the auxiliary field must be Gaussian at all times. This assumption, present in many of the approximate theories, has been critically assessed by Yeung Oono, and Shinozaki [6]; they showed that the auxiliary-field distribution measured directly from numerical simulations is not exactly Gaussian. Hence, the advantage of the approach outlined above is that the auxiliary-field distribution may take any form.

Secondly, it has been proposed by several authors [10,11] that OJK theory is exact in an infinite number of dimensions. Determining the dimensional dependence of the correction term allows this hypothesis to be examined.

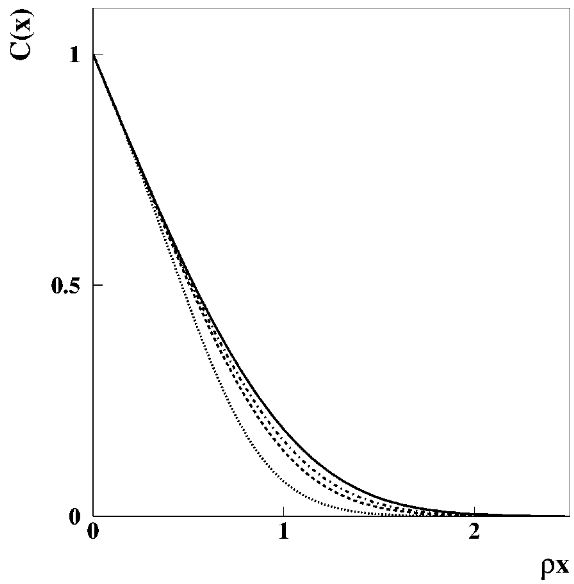

FIG. 2. Comparison of the OJK and perturbed correlation functions, shown for $d=2$, 3, and 4. The solid line is the OJK result, which is independent of dimension. As the dimension increases, the size of the perturbation decreases.

The main result of this paper is that the first order correction term to the correlation function is $O(1/d^2)$, which gives further confidence to the assertion that OJK theory becomes exact in the large-d limit. This may be compared to the calculation of Liu and Mazenko [11], in which they made a perturbative expansion near $d=1$ and $d=\infty$ about the approximate theory developed by Mazenko [15]. At large d, the first order correction term in their expansion was found to be $O(1/d)$. The relative sizes of these corrections are consistent with numerical simulations [4], which show that the OJK result provides a marginally more accurate prediction of simulation data for the pair correlation function than the more sophisticated approach of Mazenko. However both results do suggest that the results of OJK are asymptotically exact in infinite-dimensional systems. We also note that in a recent work, Mazenko [8] developed a perturbation expansion in the cumulants, which at zeroth order recovers the OJK result. At second order, however, significant deviations from the OJK theory are obtained at large d, in contrast to the $O(1/d^2)$ correction obtained here.

The two-time correction term to the correlation function is given by Eq. (88). In the limit $t_1\gg t_2$ [on comparing Eqs. (94) and (96)], we find that the correction term has exactly the same form as the zero order result, and the expected OJK result, $\bar{\lambda}=d/2$, is recovered.

If we examine the leading order correction at equal times, given by Eq. (91), the first observation is that this function is odd in r, as is the zero order term. Porod's law [12] is therefore obeyed; the $O(r)$ term in the expansion [Eq. (93)] modifies the amplitude of the large-k tail, which is proportional to the density of defects [1,12,16]. The absence of an r^2 term ensures that the correction term satisfies the Tomita sum rule [13], $\int_0^\infty dk[k^{d+1}S(k)-A]=0$, where A is the amplitude of the Porod tail. We also note that the perturbation has a negligible effect on the large-distance behavior of the correlation function, see Fig. 1.

Figure 2 demonstrates how the perturbation modifies the OJK result, plotting C_0+C_1 for $d=2$, 3, and 4. Although this calculation is only valid at large d, this graph demonstrates that the correction term will have the effect of lowering the OJK result. This is exactly as expected, since Humayun and Bray [4] showed, by comparing simulation results with OJK theory, that while the OJK result provides an accurate prediction for initial conditions with short-range correlations, the theoretical result is slightly higher than the simulation data. However, it has also been demonstrated that when long-range correlation are present in the initial conditions, the OJK results are no longer satisfactory [4]. This suggests that a possible extension to this work could be to consider the effects of long-range initial conditions on the calculation.

The main limitation of this calculation is that to retrieve the full evolution equation for the auxiliary field [Eq. (9)], we need to set the perturbation parameter λ to 1. This means that the calculation of the entire correction term requires a sum over all orders in λ. However, due to the complexity of the present calculation, I have been unable to evaluate the correction terms at higher orders in λ. In principle the sum of the higher order terms could alter the d dependence of the correction term, but the result remains a strong indication that OJK theory becomes exact in an infinite-dimensional system.

Finally, since we noted that the distribution of the auxiliary field is non-Gaussian, it is of interest to consider the exact form of this distribution. This can be calculated as follows; the distribution can be written in the following form, $P(x)=\langle \delta[x-m(\mathbf{r},t)]\rangle$, this may be expanded in λ using $m(\mathbf{r},t)=m_0(\mathbf{r},t)+\lambda m_1(\mathbf{r},t)+O(\lambda^2)$, to give $P(x)=\langle \delta[x-m_0(\mathbf{r},t)]\rangle-\lambda\langle m_1(\mathbf{r},t)\,(d/dx)\,\delta[x-m_0(\mathbf{r},t)]\rangle$. The first term on the right-hand side reduces to the expected Gaussian distribution; the first order term in λ can be evaluated using a similar method to the correlation function calculation, i.e., by inserting the expression for $m_1(\mathbf{r},t)$, multiplying by the relevant probability distribution function, and completing the integrals. However, we will leave this question to future work.

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APPENDIX A: EVALUATION OF THE JOINT PROBABILITY DISTRIBUTION

In this appendix the joint probability distribution is explicitly calculated. It is defined by

$$P(\tilde{\mathbf{m}})=\frac{1}{(2\pi)^{(d+3)/2}(\det A^{-1})^{1/2}}\exp\left[-\frac{1}{2}\tilde{m}_iA_{ij}\tilde{m}_j\right], \tag{A1}$$

where $A_{ij}^{-1}=\langle\tilde{m}_i\tilde{m}_j\rangle$ and the vector $\tilde{\mathbf{m}}$ is defined in Eq. (32). The first step is to calculate all the correlators which define the elements of the matrix A^{-1}; we can then find the inverse of this matrix (A_{ij}) and the determinant $\det A^{-1}$.

1. Calculation of the correlators

In this section we evaluate all the elements of the matrix $A_{ij}^{-1}=\langle\tilde{m}_i\tilde{m}_j\rangle$; for this we require the following correlators: $\langle m_0(\mathbf{x},t)m_0(\mathbf{x}',t')\rangle$, $\langle m_0(\mathbf{x},t)\,(\partial/\partial x_i')\,m_0(\mathbf{x}',t')\rangle$, and $\langle(\partial/\partial x_i)\,m_0(\mathbf{x},t)\,(\partial/\partial x_j)\,m_0(\mathbf{x},t)\rangle$.

Substituting Eq. (20) into Eq. (18) gives an expression for the auxiliary field $m_0(\mathbf{x},t)$:

$$m_0(\mathbf{x},t)=\frac{1}{(4\pi Dt)^{d/2}}\int d^2\mu\; m_0(\boldsymbol{\mu},0)\exp\left[-\frac{(\mathbf{x}-\boldsymbol{\mu})^2}{4Dt}\right], \tag{A2}$$

and substituting this expression into the correlator $\langle m_0(\mathbf{x},t)m_0(\mathbf{x}',t')\rangle$ gives

$$\begin{aligned}\langle m_0(\mathbf{x},t)m_0(\mathbf{x}',t')\rangle &= \frac{1}{(4\pi D)^d(tt')^{d/2}}\int d^d\mu\int d^d\eta\Delta\,\delta(\boldsymbol{\mu}-\boldsymbol{\eta})\\ &\quad\times\exp\left[-\frac{(\mathbf{x}-\boldsymbol{\mu})^2}{4Dt}-\frac{(\mathbf{x}'-\boldsymbol{\eta})^2}{4Dt'}\right],\end{aligned} \tag{A3}$$

where we have already applied the conventional Gaussian initial conditions $\langle m_0(\mathbf{x}+\mathbf{r},0)m_0(\mathbf{x},0)\rangle=\Delta\,\delta(\mathbf{r})$.

This integral may be evaluated by completing the square in the exponent and making a change of variables $\boldsymbol{\mu}'=\boldsymbol{\mu}-(t'\mathbf{x}+t\mathbf{x}')/(t+t')$, leaving a simple Gaussian integral which, once completed, gives

$$\langle m_0(\mathbf{x},t)m_0(\mathbf{x}',t')\rangle=\frac{\Delta}{[4\pi D(t+t')]^{d/2}}\exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{4D(t+t')}\right]. \tag{A4}$$

The remaining correlators are easily obtained by differentiating Eq. (A4); we obtain

$$\left\langle m_0(\mathbf{x},t)\frac{\partial}{\partial x_i'}m_0(\mathbf{x}',t')\right\rangle=\frac{(x_i-x_i')}{2D(t+t')}\langle m_0(\mathbf{x},t)m_0(\mathbf{x}',t')\rangle, \tag{A5}$$

$$\left\langle\frac{\partial}{\partial x_i}m_0(\mathbf{x},t)\frac{\partial}{\partial x_j}m_0(\mathbf{x},t)\right\rangle=\frac{\delta_{ij}}{4Dt}\langle m_0(\mathbf{x},t)m_0(\mathbf{x},t)\rangle. \tag{A6}$$

Substituting Eqs. (A4), (A5) and (A6) into the definition of A^{-1} ($A_{ij}^{-1}=\langle\tilde{m}_i\tilde{m}_j\rangle$), we find

$$A^{-1}=\left(\frac{\lambda}{z}\right)\begin{pmatrix} 1 & & 0 & \vdots & \vdots & \vdots \\ & \ddots & & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ 0 & & 1 & \vdots & \vdots & \vdots \\ \cdots & \mathbf{a}^T & \cdots & z\lambda_1 & zw & zy \\ \cdots & \mathbf{b}^T & \cdots & zw & z\lambda_2 & zv \\ \cdots & \mathbf{c}^T & \cdots & zy & zv & z\lambda_3 \end{pmatrix}, \tag{A7}$$

where

$$z=4D\tau,$$

$$\lambda=\frac{\Delta}{(8\pi D\tau)^{d/2}},\quad w=\left(\frac{2\tau}{\tau+t_1}\right)^{d/2}\exp\left[-\frac{(\mathbf{x}+\mathbf{r}-\boldsymbol{\nu})^2}{4D(t_1+\tau)}\right],$$

$$\lambda_1=\left(\frac{\tau}{t_1}\right)^{d/2},\quad y=\left(\frac{2\tau}{t_1+t_2}\right)^{d/2}\exp\left[-\frac{\mathbf{r}^2}{4D(t_1+t_2)}\right],$$

$$\lambda_2=1,\quad v=\left(\frac{2\tau}{\tau+t_2}\right)^{d/2}\exp\left[-\frac{(\mathbf{x}-\boldsymbol{\nu})^2}{4D(t_2+\tau)}\right], \tag{A8}$$

$$\lambda_3=\left(\frac{\tau}{t_2}\right)^{d/2},$$

$$\mathbf{a}=(\mathbf{x}+\mathbf{r}-\boldsymbol{\mu})\left(\frac{2\tau}{\tau+t_1}\right)^{(d+2)/2}\exp\left[-\frac{(\mathbf{x}+\mathbf{r}-\boldsymbol{\mu})^2}{4D(t_1+\tau)}\right],$$

$$\mathbf{b}=(\boldsymbol{\nu}-\boldsymbol{\mu})\exp\left[-\frac{(\boldsymbol{\nu}-\boldsymbol{\mu})^2}{8D\tau}\right],$$

$$\mathbf{c}=(\mathbf{x}-\boldsymbol{\mu})\left(\frac{2\tau}{\tau+t_2}\right)^{(d+2)/2}\exp\left[-\frac{(\mathbf{x}-\boldsymbol{\mu})^2}{4D(t_2+\tau)}\right].$$

2. Calculation of A

The elements of the matrix A are calculated by constructing the adjoint and determinant of the inverse, since $A=\mathrm{Adj}(A^{-1})/\det(A^{-1})$. Let the elements of the adjoint be defined by

$$\mathrm{Adj}(A^{-1})=\begin{pmatrix} & & & \vdots & \vdots & \vdots \\ & \theta_{ij} & & \boldsymbol{\xi} & \boldsymbol{\eta} & \boldsymbol{\zeta} \\ & & & \vdots & \vdots & \vdots \\ \cdots & \boldsymbol{\xi}^T & \cdots & p & s & t \\ \cdots & \boldsymbol{\eta}^T & \cdots & s & q & u \\ \cdots & \boldsymbol{\zeta}^T & \cdots & t & u & r \end{pmatrix}. \tag{A9}$$

The elements $p,q,r,\ldots,u$ can then be calculated directly using a formula for the determinant of a $(d+2)\times(d+2)$ matrix of the form

$$B=\begin{pmatrix} 1 & & 0 & \vdots & \vdots \\ & \ddots & & \mathbf{a}' & \mathbf{b}' \\ 0 & & 1 & \vdots & \vdots \\ \cdots & \mathbf{c}'^T & \cdots & p' & q' \\ \cdots & \mathbf{d}'^T & \cdots & r' & s' \end{pmatrix}, \tag{A10}$$

which is given by

$$\det B=(p'-\mathbf{a}'\cdot\mathbf{c}')(s'-\mathbf{b}'\cdot\mathbf{d}')-(q'-\mathbf{b}'\cdot\mathbf{c}')(r'-\mathbf{a}'\cdot\mathbf{d}'). \tag{A11}$$

We derive this result by contracting the free indices in the equation

$$\det B=\sum_{ij\ldots xyz}\epsilon_{ij\cdots xyz}B_{1i}B_{2j}\cdots B_{d+2z}, \tag{A12}$$

where $\epsilon_{ij\ldots xyz}$ is a $d+2$ anisotropic tensor, which takes the value 1 if $(ij\cdots xyz)$ is an even permutation of $[123\cdots(d+2)]$, -1 for an odd permutation, and zero otherwise.

First consider the sum over the index i: B_{1i} is only nonzero if $i=1, d+1$, or $d+2$, ($B_{11}=1$, $B_{1d+1}=a_1'$ and $B_{1d+2}=b_1'$), which implies that Eq. (A12) may be written as

$$\det B=\sum_{jk\cdots xyz}\epsilon_{1j\ldots xyz}B_{2j}\cdots B_{d+2z}+D_1, \tag{A13}$$

where

$$D_1=\sum_{jk\cdots xyz}(\epsilon_{d+1jk\ldots xyz}a_1'B_{2j}\cdots B_{d+2z}+\epsilon_{d+2jk\ldots xyz}b_1'B_{2j}\cdots B_{d+2z}). \tag{A14}$$

We now complete the sum over j in Eq. (A14) by noting that the first term on the right-hand side of this expression will only be nonzero if $j=2$ or $j=d+2$, whereas the second term will only make a nonzero contribution if $j=2$ or $j=d+1$, giving

$$D_1=\sum_{k\cdots xyz}(\epsilon_{d+1d+2k\ldots xyz}a_1'b_2'B_{3k}\cdots B_{d+2z}+\epsilon_{d+2d+1k\ldots xyz}b_1'a_2'B_{3k}\cdots B_{d+2z}+\epsilon_{d+12k\ldots xyz}a_1'B_{3k}\cdots B_{d+2z}+\epsilon_{d+2k\ldots xyz}b_1'B_{3k}\cdots B_{d+2z}). \tag{A15}$$

We can now complete the sum over all the remaining indices for the first two terms on the right-hand side of Eq. (A15), since for a nonzero contribution we must have $k=3$, $l=4,\ldots,x=d$ and either $y=1$ and $z=2$, or $y=2$ and $z=1$. After completing the sum over k on the remaining right-hand side terms, Eq. (A15) gives

$$D_1=a_1'c_1'b_2'd_2'+b_1'd_1'a_2'c_2'-a_1'd_1'b_2'c_2'-b_1'c_1'a_2'd_2'+\sum_{l\,\ldots\,xyz}(\epsilon_{d+12d+2l\,\ldots\,xyz}a_1'b_3'B_{4l}\cdots B_{d+2z}$$
$$+\epsilon_{d+23d+1l\,\ldots\,xyz}b_1'a_3'B_{4l}\cdots B_{d+2z}+\epsilon_{d+123l\,\ldots\,xyz}a_1'B_{4l}\cdots B_{d+2z}+\epsilon_{d+223l\,\ldots\,xyz}b_1'B_{4l}\cdots B_{d+2z}). \quad \text{(A16)}$$

We can use this method repeatedly to sum over all the free indices in Eq. (A16), eventually giving

$$D_1=\sum_{j=2}^{d}(a_1'c_1'b_j'd_j'+b_1'd_1'a_j'c_j'-a_1'd_1'b_j'c_j'-b_1'c_1'a_j'd_j')$$
$$-sa_1'c_1'+qa_1'd_1'-pb_1'd_1'+rb_1'c_1'. \quad \text{(A17)}$$

Substituting Eq. (A17) back into Eq. (A13), we obtain

$$\det B=\sum_{jk\,\ldots\,xyz}\epsilon_{1j\,\ldots\,xyz}B_{2j}\cdots B_{d+2z}-sa_1'c_1'+qa_1'd_1'$$
$$-pb_1'd_1'+rb_1'c_1'+\sum_{j=2}^{d}(a_1'c_1'b_j'd_j'+b_1'd_1'a_j'c_j'$$
$$-a_1'd_1'b_j'c_j'-b_1'c_1'a_j'd_j'). \quad \text{(A18)}$$

We can now repeat this entire procedure to complete the sums over the indices $j\,\ldots\,x$, obtaining

$$\det B=\sum_{yz}\epsilon_{12\,\ldots\,dyz}B_{d+1y}B_{d+2z}-s\mathbf{a}'\cdot\mathbf{c}'+q\mathbf{a}'\cdot\mathbf{d}'-p\mathbf{b}'\cdot\mathbf{d}'$$
$$+r\mathbf{b}'\cdot\mathbf{c}'+\sum_{i=1}^{d}\sum_{j>i}^{d}(a_i'c_i'b_j'd_j'+b_i'd_i'a_j'c_j'$$
$$-a_i'd_i'b_j'c_j'-b_i'c_i'a_j'd_j'). \quad \text{(A19)}$$

Two of these terms simplify further; we find

$$\sum_{yz}\epsilon_{12\,\ldots\,dyz}B_{d+1y}B_{d+2z}=ps-qr,$$

$$\sum_{i=1}^{d}\sum_{j>i}^{d}(a_i'c_i'b_j'd_j'+b_i'd_i'a_j'c_j'-a_i'd_i'b_j'c_j'-b_i'c_i'a_j'd_j')$$
$$=(\mathbf{a}'\cdot\mathbf{c}')(\mathbf{b}'\cdot\mathbf{d}')-(\mathbf{a}'\cdot\mathbf{d}')(\mathbf{b}'\cdot\mathbf{c}'),$$

and therefore Eq. (A19) finally reduces to

$$\det B=(p'-\mathbf{a}'\cdot\mathbf{c}')(s'-\mathbf{b}'\cdot\mathbf{d}')-(q'-\mathbf{b}'\cdot\mathbf{c}')(r'-\mathbf{a}'\cdot\mathbf{d}'). \quad \text{(A20)}$$

Having calculated the determinant of B, we can use this result to evaluate the elements $p,q,r,\ldots,u$ of the adjoint matrix [Eq. (A9)]. We obtain

$$p=\left(\frac{\lambda}{z}\right)^{d+2}[(z\lambda_2-\mathbf{b}^2)(z\lambda_3-\mathbf{c}^2)-(zv-\mathbf{b}\cdot\mathbf{c})^2], \quad \text{(A21)}$$

$$q=\left(\frac{\lambda}{z}\right)^{d+2}[(z\lambda_1-\mathbf{a}^2)(z\lambda_3-\mathbf{c}^2)-(zy-\mathbf{a}\cdot\mathbf{c})^2], \quad \text{(A22)}$$

$$r=\left(\frac{\lambda}{z}\right)^{d+2}[(z\lambda_1-\mathbf{a}^2)(z\lambda_2-\mathbf{b}^2)-(zw-\mathbf{a}\cdot\mathbf{b})^2], \quad \text{(A23)}$$

$$s=\left(\frac{\lambda}{z}\right)^{d+2}[(zv-\mathbf{b}\cdot\mathbf{c})(zy-\mathbf{a}\cdot\mathbf{c})-(zw-\mathbf{a}\cdot\mathbf{b})(z\lambda_3-\mathbf{c}^2)], \quad \text{(A24)}$$

$$t=\left(\frac{\lambda}{z}\right)^{d+2}[(zw-\mathbf{a}\cdot\mathbf{b})(zv-\mathbf{b}\cdot\mathbf{c})-(z\lambda_2-\mathbf{b}^2)(zy-\mathbf{a}\cdot\mathbf{c})], \quad \text{(A25)}$$

$$u=\left(\frac{\lambda}{z}\right)^{d+2}[(zw-\mathbf{a}\cdot\mathbf{b})(zy-\mathbf{a}\cdot\mathbf{c})-(z\lambda_1-\mathbf{a}^2)(zv-\mathbf{b}\cdot\mathbf{c})]. \quad \text{(A26)}$$

The remaining elements of the adjoint are evaluated directly by a componentwise expansion of the equation $A^{-1}\mathrm{Adj}(A^{-1})=\det(A^{-1})I$, giving the expressions

$$\boldsymbol{\xi}=-(p\mathbf{a}+s\mathbf{b}+t\mathbf{c}), \quad \text{(A27)}$$

$$\boldsymbol{\eta}=-(s\mathbf{a}+q\mathbf{b}+u\mathbf{c}), \quad \text{(A28)}$$

$$\boldsymbol{\zeta}=-(t\mathbf{a}+u\mathbf{b}+r\mathbf{c}), \quad \text{(A29)}$$

$$\theta_{ij}=\frac{z}{\lambda}(\det A^{-1})\,\delta_{ij}-a_i\xi_j-b_i\eta_j-c_i\zeta_j. \quad \text{(A30)}$$

Now that we have evaluated all the components of the adjoint, we can evaluate the expression for the joint probability distribution, $P(\tilde{\mathbf{m}})$, in terms of these variables. Substituting Eq. (A9) into Eq. (A1) and using $A=\mathrm{Adj}(A^{-1})/\det(A^{-1})$, we find

$$P(\tilde{\mathbf{m}})=\frac{1}{(2\pi)^{(d+3)/2}(\det A^{-1})^{1/2}}\exp\left[-\frac{F(\tilde{\mathbf{m}})}{2\det A^{-1}}\right], \quad \text{(A31)}$$

where

$$F(\tilde{\mathbf{m}})=[m_k'\theta_{kl}m_l'+2\boldsymbol{\xi}\cdot\mathbf{m}'m(1)+2\boldsymbol{\eta}\cdot\mathbf{m}'m(2)$$
$$+2\boldsymbol{\zeta}\cdot\mathbf{m}'m(3)+pm(1)^2+qm(2)^2+rm(3)^2$$
$$+2sm(1)m(2)+2tm(1)m(3)+2um(2)m(3)]. \quad \text{(A32)}$$

3. Identities relating the components of the adjoint of A^{-1} to the determinant of A^{-1}

In the calculation described in this paper, there are several expressions which are frequently used to simplify the algebra. These are all derived directly from the componentwise expansion of the equation $A^{-1}\mathrm{Adj}(A^{-1})=\det A^{-1}I$, and are listed here for ease of reference.

$$p(z\lambda_1-\mathbf{a}^2)+s(zw-\mathbf{a}\cdot\mathbf{b})+t(zy-\mathbf{a}\cdot\mathbf{c})=\frac{z}{\lambda}\det A^{-1}, \tag{A33}$$

$$s(z\lambda_1-\mathbf{a}^2)+q(zw-\mathbf{a}\cdot\mathbf{b})+u(zy-\mathbf{a}\cdot\mathbf{c})=0, \tag{A34}$$

$$t(z\lambda_1-\mathbf{a}^2)+u(zw-\mathbf{a}\cdot\mathbf{b})+r(zy-\mathbf{a}\cdot\mathbf{c})=0, \tag{A35}$$

$$p(zw-\mathbf{a}\cdot\mathbf{b})+s(z\lambda_2-\mathbf{b}^2)+t(zv-\mathbf{b}\cdot\mathbf{c})=0, \tag{A36}$$

$$s(zw-\mathbf{a}\cdot\mathbf{b})+q(z\lambda_2-\mathbf{b}^2)+u(zv-\mathbf{b}\cdot\mathbf{c})=\frac{z}{\lambda}\det A^{-1}, \tag{A37}$$

$$t(zw-\mathbf{a}\cdot\mathbf{b})+u(z\lambda_2-\mathbf{b}^2)+r(zv-\mathbf{b}\cdot\mathbf{c})=0, \tag{A38}$$

$$p(zy-\mathbf{a}\cdot\mathbf{c})+s(zv-\mathbf{b}\cdot\mathbf{c})+t(z\lambda_3-\mathbf{c}^2)=0, \tag{A39}$$

$$s(zy-\mathbf{a}\cdot\mathbf{c})+q(zv-\mathbf{b}\cdot\mathbf{c})+u(z\lambda_3-\mathbf{c}^2)=0, \tag{A40}$$

$$t(zy-\mathbf{a}\cdot\mathbf{c})+u(zv-\mathbf{b}\cdot\mathbf{c})+r(z\lambda_3-\mathbf{c}^2)=\frac{z}{\lambda}\det A^{-1}. \tag{A41}$$

In addition, by using the definitions of $p,q,\ldots,u$ [from Eqs. (A21)–(A26)] and Eqs. (A33), (A34), (A37), (A39), and (A40), we can see that

$$(pq-s^2)=\left(\frac{\lambda}{z}\right)^{d+1}\det A^{-1}(z\lambda_3-\mathbf{c}^2), \tag{A42}$$

$$(qr-u^2)=\left(\frac{\lambda}{z}\right)^{d+1}\det A^{-1}(z\lambda_1-\mathbf{a}^2), \tag{A43}$$

$$(us-qt)=\left(\frac{\lambda}{z}\right)^{d+1}\det A^{-1}(zy-\mathbf{a}\cdot\mathbf{c}). \tag{A44}$$

APPENDIX B: CALCULATION OF THE INVERSE AND DETERMINANT OF Ω

In this appendix, the expressions for the determinant and inverse of the matrix Ω, which are required for the completion of the integrals in Eqs. (47)–(50), are calculated. The matrix $\Omega(\tilde{u},\tilde{v})$ is defined by

$$\Omega_{ij}(\tilde{u},\tilde{v})=\frac{1}{\det A^{-1}}\left(\theta_{ij}-\frac{\eta_i\eta_j}{q}+\frac{(pq-s^2)}{q}h_ih_j(\tilde{u}^2-1)\right)+2\tilde{v}\,\delta_{ij}, \tag{B1}$$

where $\mathbf{h}$, η, and θ are defined by Eqs. (38), (A27), and (A30), respectively. Substituting for $\mathbf{h}$, η and θ into Eq. (B1), and using Eqs. (A22), (A29), and (A42)–(A44), we find the expression for $\Omega(\tilde{u},\tilde{v})$ reduces to

$$\Omega_{ij}(\tilde{u},\tilde{v})=\frac{z}{\lambda}[\Lambda\,\delta_{ij}+\tilde{\mu}a_ia_j+\tilde{\xi}(a_ic_j+c_ia_j)+\tilde{\nu}c_ic_j], \tag{B2}$$

where

$$\Lambda=1+2\lambda\tilde{v}/z, \tag{B3}$$

$$\tilde{\mu}=\frac{\tilde{u}^2}{\tilde{q}}(z\lambda_3-\mathbf{c}^2), \tag{B4}$$

$$\tilde{\xi}=-\frac{\tilde{u}^2}{\tilde{q}}(zy-\mathbf{a}\cdot\mathbf{c}), \tag{B5}$$

$$\tilde{\nu}=\frac{\tilde{u}^2}{\tilde{q}}(z\lambda_1-\mathbf{a}^2)+\frac{1-\tilde{u}^2}{z\lambda_3-\mathbf{c}^2}, \tag{B6}$$

$$\tilde{q}=\left(\frac{z}{\lambda}\right)^{d+2}q=(z\lambda_1-\mathbf{a}^2)(z\lambda_3-\mathbf{c}^2)-(zy-\mathbf{a}\cdot\mathbf{c})^2. \tag{B7}$$

1. Determinant of $\Omega(\tilde{u},\tilde{v})$

The determinant of the matrix $\Omega(\tilde{u},\tilde{v})$ is calculated by evaluating the product of its d eigenvalues. Any vector orthogonal to both $\mathbf{a}$ and $\mathbf{c}$ will have eigenvalue $z\Lambda/\lambda$, so we need only calculate the two remaining eigenvalues, which are associated with the eigenvectors which lie in the plane spanned by $\mathbf{a}$ and $\mathbf{c}$. The eigenvalue equation for these two may be written as

$$\Omega_{ij}(a_j+\gamma c_j)=\beta(a_i+\gamma c_i). \tag{B8}$$

We can substitute for $\Omega(\tilde{u},\tilde{v})$ from Eq. (B1) and complete the sum over the index j; by equating the coefficients of a_i and c_i we then obtain two simultaneous equations for β and γ, given by

$$\beta=\frac{z}{\lambda}[\Lambda+\tilde{\mu}\mathbf{a}^2+\tilde{\xi}\mathbf{a}\cdot\mathbf{c}+\gamma(\tilde{\mu}\mathbf{a}\cdot\mathbf{c}+\tilde{\xi}\mathbf{c}^2)], \tag{B9}$$

$$\beta\gamma=\frac{z}{\lambda}[\tilde{\xi}\mathbf{a}^2+\tilde{\nu}\mathbf{a}\cdot\mathbf{c}+\gamma(\Lambda+\tilde{\xi}\mathbf{a}\cdot\mathbf{c}+\tilde{\nu}\mathbf{c}^2)]. \tag{B10}$$

On eliminating γ between these two equations, we have a quadratic equation in β, from which we extract the product of the two roots of the quadratic, $\beta_\pm$. This product is given by

$$\beta_+\beta_-=\left(\frac{z}{\lambda}\right)^2\Delta(\tilde{u},\tilde{v}), \tag{B11}$$

where

$$\Delta(\tilde{u},\tilde{v})=(\Lambda+\tilde{\xi}\mathbf{a}\cdot\mathbf{c}+\tilde{\nu}\mathbf{c}^2)(\Lambda+\tilde{\mu}\mathbf{a}^2+\tilde{\xi}\mathbf{a}\cdot\mathbf{c})-(\tilde{\xi}\mathbf{a}^2+\tilde{\nu}\mathbf{a}\cdot\mathbf{c})\times(\tilde{\mu}\mathbf{a}\cdot\mathbf{c}+\tilde{\xi}\mathbf{c}^2). \tag{B12}$$

To simplify this expression we expand the brackets on the right-hand side, and use Eqs. (B4)–(B7); after some algebra, Eq. (B12) reduces to

$$\Delta(\tilde{u},\tilde{v})=\Lambda\left(\frac{\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2}{z\lambda_3-\mathbf{c}^2}\right)-\frac{\Lambda z\lambda_3\tilde{u}^2}{(z\lambda_3-\mathbf{c}^2)}+\frac{\Lambda z^2\tilde{u}^2}{\tilde{q}}(\lambda_1\lambda_3-y^2)-\frac{(\Lambda-1)\tilde{u}^2}{\tilde{q}}(\mathbf{a}^2\mathbf{c}^2-\mathbf{a}\cdot\mathbf{c}^2). \tag{B13}$$

We recall that the remaining eigenvectors all have the eigenvalue $z\Lambda/\lambda$, and therefore the product of all the eigenvalues, which is equal to the determinant, is given by

$$\det\Omega=\Lambda^{d-2}\left(\frac{z}{\lambda}\right)^d\Delta(\tilde{u},\tilde{v}). \tag{B14}$$

2. Inverse of $\Omega(\tilde{u},\tilde{v})$

We begin to find the inverse of $\Omega(\tilde{u},\tilde{v})$ by defining the variables $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ by the equation

$$\Omega_{ij}^{-1}=\frac{\lambda}{z}(\mathcal{A}\delta_{ij}+\mathcal{B}a_ia_j+\mathcal{C}(a_ic_j+c_ia_j)+\mathcal{D}c_ic_j). \tag{B15}$$

This must satisfy the identity $\Omega_{ij}\Omega_{jk}^{-1}=\delta_{ik}$. Hence, by expanding this equation and equating coefficients, we obtain the set of simultaneous equations given below:

$$O(1),\quad \mathcal{A}\Lambda=1, \tag{B16}$$

$$O(a_ia_k),\quad \mathcal{A}\tilde{\mu}+\mathcal{B}(\Lambda+\tilde{\mu}\mathbf{a}^2+\tilde{\xi}\mathbf{a}\cdot\mathbf{c})+\mathcal{C}(\tilde{\mu}\mathbf{a}\cdot\mathbf{c}+\tilde{\xi}\mathbf{c}^2)=0, \tag{B17}$$

$$O(a_ic_k),\quad \mathcal{A}\tilde{\xi}+\mathcal{C}(\Lambda+\tilde{\mu}\mathbf{a}^2+\tilde{\xi}\mathbf{a}\cdot\mathbf{c})+\mathcal{D}(\tilde{\mu}\mathbf{a}\cdot\mathbf{c}+\tilde{\xi}\mathbf{c}^2)=0, \tag{B18}$$

$$O(c_ia_k),\quad \mathcal{A}\tilde{\xi}+\mathcal{B}(\tilde{\xi}\mathbf{a}^2+\tilde{\nu}\mathbf{a}\cdot\mathbf{c})+\mathcal{C}(\Lambda+\tilde{\xi}\mathbf{a}\cdot\mathbf{c}+\tilde{\nu}\mathbf{c}^2)=0, \tag{B19}$$

$$O(c_ic_k),\quad \mathcal{A}\tilde{\nu}+\mathcal{C}(\tilde{\xi}\mathbf{a}^2+\tilde{\nu}\mathbf{a}\cdot\mathbf{c})+\mathcal{D}(\Lambda+\tilde{\xi}\mathbf{a}\cdot\mathbf{c}+\tilde{\nu}\mathbf{c}^2)=0. \tag{B20}$$

Since we have one more equation than we require to determine the solutions for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, we discard one equation and check for consistency later. Solving Eqs. (B16), (B17), (B18), and (B20) simultaneously, and using Eqs. (B7) and (B13) to simplify the results, we find, after some algebra, that

$$\mathcal{A}=\Lambda^{-1}, \tag{B21}$$

$$\mathcal{B}=-\frac{\tilde{u}^2}{\Lambda\tilde{q}\Delta}[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2], \tag{B22}$$

$$\mathcal{C}=\frac{\tilde{u}^2}{\Lambda\tilde{q}\Delta}[\Lambda zy-(\Lambda-1)\mathbf{a}\cdot\mathbf{c}], \tag{B23}$$

$$\mathcal{D}=-\frac{\tilde{u}^2}{\Lambda\tilde{q}\Delta}[\Lambda z\lambda_1-(\Lambda-1)\mathbf{a}^2]+\frac{(\tilde{u}^2-1)}{\Delta(z\lambda_3-\mathbf{c}^2)}. \tag{B24}$$

Substituting these results back into Eq. (B15), we see that the inverse of $\Omega(\tilde{u},\tilde{v})$ is given by

$$\Omega_{ij}^{-1}(\tilde{u},\tilde{v})=\frac{\lambda}{\Lambda z}\Bigg[\delta_{ij}+\frac{\Lambda(\tilde{u}^2-1)}{\Delta(z\lambda_3-\mathbf{c}^2)}+\frac{\tilde{u}^2}{\tilde{q}\Delta}[\Lambda zy-(\Lambda-1)\mathbf{a}\cdot\mathbf{c}](a_ic_j+c_ia_j)-\frac{\tilde{u}^2}{\tilde{q}\Delta}\{[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]a_ia_j+[\Lambda z\lambda_1-(\Lambda-1)\mathbf{a}^2]c_ic_j\}\Bigg]. \tag{B25}$$

This expression may be simplified considerably by the introduction of a new variable. If we define **k** by the equation

$$k_i=[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]a_i-[\Lambda zy-(\Lambda-1)\mathbf{a}\cdot\mathbf{c}]c_j, \tag{B26}$$

then Eq. (B25) may be rewritten as

$$\Omega_{ij}^{-1}=\frac{\lambda}{\Lambda z}\left(\delta_{ij}-\frac{c_ic_j}{[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}-\frac{\tilde{u}^2k_ik_j}{\tilde{q}\Delta[\Lambda z\lambda_3-(\Lambda-1]\mathbf{c}^2)}\right). \tag{B27}$$

APPENDIX C: CONTRACTION OVER THE i AND j INDICES

In this appendix we calculate a nonindexed expression for

$$\left.\frac{\partial^2 I_{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}=\sum_{n=1}^{4}\left.\frac{\partial^2 I_n^{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} \tag{C1}$$

where:

$$\left.\frac{\partial^2 I_1^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = \frac{\delta_{ij}}{d\pi(z\lambda_3 - \mathbf{c}^2)[\tilde{q}\Delta(1,0)]^{1/2}} \left.\frac{\partial^2 \tilde{s}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}, \tag{C2}$$

$$\left.\frac{\partial^2 I_2^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\frac{1}{\pi(z\lambda_3 - \mathbf{c}^2)} \left.\frac{\partial^2 \tilde{s}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} \int_0^\infty d\tilde{v} \frac{\Omega_{ij}^{-1}(1,\tilde{v})}{\Lambda^{(d-2)/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}}, \tag{C3}$$

$$\left.\frac{\partial^2 I_3^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = \frac{\mathbf{k}(0).\mathbf{c}\delta_{ij}}{d\pi\lambda_3(z\lambda_3 - \mathbf{c}^2)[\tilde{q}\Delta(1,0)]^{1/2}} \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}}, \tag{C4}$$

$$\left.\frac{\partial^2 I_4^{ij}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\frac{z}{\pi(z\lambda_3 - \mathbf{c}^2)} \int_0^\infty \frac{d\tilde{v}}{\Lambda^{d/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}} \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} \times \left(\frac{\mathbf{k}(\tilde{v}).\mathbf{c}\Omega_{ij}^{-1}(1,\tilde{v})}{[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]} + \frac{\lambda(z\lambda_3 - \mathbf{c}^2)[k_i(\tilde{v})c_j + c_i k_j(\tilde{v})]}{z[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]^2} \right), \tag{C5}$$

and $\tilde{s} = (z/\lambda)^{d+2} s$, s being defined by Eq. (A24).

To obtain such an expression, we first evaluate the sum of the four expressions given by Eqs. (C2)–(C5), and then contract this expression over the free indices i and j.

To simplify these expressions we need to calculate the second derivative of $\tilde{s}$. Using $\partial^2 \mathbf{b}/\partial \nu_i \partial \nu_j|_{\boldsymbol{\mu}} = 0$ [from Eq. (A8)], and Eq. (A24), we find that

$$\left.\frac{\partial^2 \tilde{s}}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = z\left((zy - \mathbf{a}\cdot\mathbf{c}) \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} - (z\lambda_3 - \mathbf{c}^2) \left.\frac{\partial w}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} \right). \tag{C6}$$

After substituting for $\mathbf{k}$ and the second derivative of $\tilde{s}$ (from equations (B26) and (C6) respectively), into equations (C2)-(C5), we may combine the resulting expressions to obtain:

$$\left.\frac{\partial^2 (I_1^{ij} + I_3^{ij})}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = \frac{z\delta_{ij}}{\pi d\lambda_3[\tilde{q}\Delta(1,0)]^{1/2}} \left(y \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} - \lambda_3 \left.\frac{\partial^2 w}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} \right), \tag{C7}$$

$$\left.\frac{\partial^2 (I_2^{ij} + I_4^{ij})}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} = -\int_0^\infty \frac{d\tilde{v}}{\pi\Lambda^{d/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}} \left[\frac{\lambda[k(\tilde{v})_i c_j + c_i k(\tilde{v})_j]}{[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]^2} \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} + z\Lambda\Omega_{ij}^{-1}(1,\tilde{v}) \left(\frac{yR}{\lambda_3} \left.\frac{\partial^2 v}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} - \left|\frac{\partial^2 w}{\partial \nu_i \partial \nu_j}\right|_{\boldsymbol{\mu}} \right) \right], \tag{C8}$$

where

$$R = \frac{\lambda_3}{y}\left(\frac{\Lambda z y - (\Lambda-1)\mathbf{a}\cdot\mathbf{c}}{\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2} \right) = 1 + \frac{(\Lambda-1)(y\mathbf{c}^2 - \lambda_3 \mathbf{a}\cdot\mathbf{c})}{y[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]}. \tag{C9}$$

Before attempting to calculate the sum in Eq. (24), we manipulate Eq. (C8) so that part of the $\tilde{v}$ integral can be completed exactly. This will simplify the algebra greatly, since the exactly integrable term in Eq. (C8) will cancel the contribution to the sum from Eq. (C7).

We define a new variable Ψ_{ij} by the equation

$$\Omega_{ij}^{-1}(1,\tilde{v})=\frac{\lambda}{\Lambda z}\left(\delta_{ij}-\frac{\Psi_{ij}}{\tilde{q}\Delta(1,\tilde{v})}\right); \tag{C10}$$

on comparing this with Eq. (B25), we find

$$\Psi_{ij}=[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]a_ia_j-[\Lambda zy-(\Lambda-1)\mathbf{a}\cdot\mathbf{c}](a_ic_j+a_jc_i)+[\Lambda z\lambda_1-(\Lambda-1)\mathbf{a}^2]c_ic_j\,. \tag{C11}$$

Substituting for R and Ω_{ij}^{-1} from Eqs. (C9) and (C10), Eq. (C8) can be written as

$$\begin{aligned}\left.\frac{\partial^2(I_2^{ij}+I_4^{ij})}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}=S-\frac{\lambda}{\pi}\int_0^\infty\frac{d\tilde{v}}{\Lambda^{d/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}}\Bigg[&\frac{[k(\tilde{v})_ic_j+c_ik(\tilde{v})_j]}{[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]^2}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}+\frac{(\Lambda-1)(y\mathbf{c}^2-\lambda_3\mathbf{a}\cdot\mathbf{c})\delta_{ij}}{\lambda_3[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\\&-\frac{\Psi_{ij}}{\tilde{q}\Delta(1,\tilde{v})}\left(\frac{yR}{\lambda_3}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}-\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\right)\Bigg],\end{aligned} \tag{C12}$$

where

$$S=-\frac{\lambda\,\delta_{ij}}{\pi}\left(\frac{y}{\lambda_3}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}-\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\right)\int_0^\infty\frac{d\tilde{v}}{\Lambda^{d/2}[\tilde{q}\Delta(1,\tilde{v})]^{1/2}}. \tag{C13}$$

We now define a second new variable Q by the equation

$$\Delta(1,\tilde{v})=\Lambda^2\Delta(1,0)[1-Q(\tilde{v})], \tag{C14}$$

and, using Eq. (B13), we find that

$$Q=\frac{\Lambda(\Lambda-1)z(\lambda_3\mathbf{a}^2-2y\mathbf{a}\cdot\mathbf{c}+\lambda_1\mathbf{c}^2)-(\Lambda-1)^2(\mathbf{a}^2\mathbf{c}^2-\mathbf{a}\cdot\mathbf{c}^2)}{\Lambda^2z^2(\lambda_1\lambda_3-y^2)}. \tag{C15}$$

On substituting for $\Delta(1,\tilde{v})$ from Eq. (C14) into Eq. (C13), and expanding the expression $(1-Q)^{-1/2}$ using the binomial theorem, we find that

$$S=\frac{-\lambda\,\delta_{ij}}{\pi[\tilde{q}\Delta(1,0)]^{1/2}}\left(\frac{y}{\lambda_3}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}-\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\right)\int_0^\infty\frac{d\tilde{v}}{\Lambda^{(d+2)/2}}\left[1+\sum_{m=1}^\infty\binom{\frac{1}{2}}{m}Q^m\right]. \tag{C16}$$

Hence, using $\int_0^\infty\Lambda^{-(d+2)/2}=z/(\lambda d)$, and inserting Eq. (C7) in the expression for S, we find that Eq. (C16) reduces to

$$S=-\frac{\lambda\,\delta_{ij}}{\pi[\tilde{q}\Delta(1,0)]^{1/2}}\left(\frac{y}{\lambda_3}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}-\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\right)\int_0^\infty\frac{d\tilde{v}}{\Lambda^{(d+2)/2}}\sum_{m=1}^\infty\binom{\frac{1}{2}}{m}Q^m-\left.\frac{\partial^2(I_1^{ij}+I_3^{ij})}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}. \tag{C17}$$

Next, substituting Eq. (C17) into Eq. (C12) and using Eq. (C1), we find that

$$\begin{aligned}\left.\frac{\partial^2 I_{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}=&-\frac{\lambda\,\delta_{ij}}{\pi[\tilde{q}\Delta(1,0)]^{1/2}}\int_0^\infty\frac{d\tilde{v}}{\Lambda^{(d+2)/2}}\sum_{m=1}^\infty\binom{\frac{1}{2}}{m}Q^m\\&-\frac{\lambda}{\pi(\tilde{q}\Delta(1,0))^{1/2}}\int_0^\infty\frac{d\tilde{v}}{\Lambda^{(d+2)/2}(1-Q)^{1/2}}\Bigg[\frac{(\Lambda-1)(y\mathbf{c}^2-\lambda_3\mathbf{a}\cdot\mathbf{c})\delta_{ij}}{\lambda_3[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}+\frac{[k(\tilde{v})_ic_j+c_ik(\tilde{v})_j]}{[\Lambda z\lambda_3-(\Lambda-1)\mathbf{c}^2]^2}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\\&-\frac{\Psi_{ij}}{\Lambda^2\tilde{q}\Delta(1,0)(1-Q)}\left(\frac{yR}{\lambda_3}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}-\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}}\right)\Bigg].\end{aligned} \tag{C18}$$

Before we start the contraction of Eq. (C18) over the indices i and j it is convenient to derive some useful results. First, we calculate the derivatives of v and w [from Eqs. (A8)]; these are given by

$$\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left[\frac{c_i c_j}{z^2 v} - \frac{v\,\delta_{ij}}{2D(t_2+\tau)}\right]_{\boldsymbol{\mu}}, \tag{C19}$$

$$\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left[\frac{a_i a_j}{z^2 w} - \frac{w\,\delta_{ij}}{2D(t_1+\tau)}\right]_{\boldsymbol{\mu}}. \tag{C20}$$

(From this point on, we will use the notation v_μ and w_μ to represent v and w evaluated $\boldsymbol{\nu}=\boldsymbol{\mu}$). Using these equations together with Eqs. (B26) and (C11), we evaluate the following expressions:

$$\delta_{ij}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu d}{2D(t_2+\tau)}\right), \tag{C21}$$

$$\delta_{ij}\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left(\frac{\mathbf{a}^2}{z^2 w_\mu} - \frac{w_\mu d}{2D(t_1+\tau)}\right), \tag{C22}$$

$$[k(\tilde{v})_i c_j + c_i k(\tilde{v})_j]\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = 2z\Lambda(\lambda_3\mathbf{a}\cdot\mathbf{c} - y\mathbf{c}^2) \times\left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu}{2D(t_2+\tau)}\right), \tag{C23}$$

$$\Psi_{ij}(\tilde{v})\,\delta_{ij} = \Lambda z(\lambda_3\mathbf{a}^2 - 2y\mathbf{a.c} + \lambda_1\mathbf{c}^2) - 2(\Lambda-1)(\mathbf{a}^2\mathbf{c}^2 - \mathbf{a}\cdot\mathbf{c}^2), \tag{C24}$$

$$\Psi_{ij}\left.\frac{\partial^2 v}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu}{2D(t_2+\tau)}\right)\Psi_{ij}\delta_{ij} - \frac{(\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2)}{z^2 v_\mu}(\mathbf{a}^2\mathbf{c}^2 - \mathbf{a}\cdot\mathbf{c}^2), \tag{C25}$$

$$\Psi_{ij}\left.\frac{\partial^2 w}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = \left(\frac{\mathbf{a}^2}{z^2 w_\mu} - \frac{w_\mu}{2D(t_1+\tau)}\right)\Psi_{ij}\delta_{ij} - \frac{[\Lambda z\lambda_1 - (\Lambda-1)\mathbf{a}^2]}{z^2 w_\mu}(\mathbf{a}^2\mathbf{c}^2 - \mathbf{a}\cdot\mathbf{c}^2). \tag{C26}$$

Finally, we can complete the contraction over i and j by substituting Eqs. (C21)–(C26), together with Eqs. (C9) and (C15)) into Eq. (C18) to give

$$\left.\frac{\partial^2 I_{ij}}{\partial\nu_i\partial\nu_j}\right|_{\boldsymbol{\mu}} = -\frac{\lambda}{z\pi}\int_0^\infty d\tilde{v}\sum_{n=1}^{6} T_n, \tag{C27}$$

where

$$T_1 = \frac{\sum_{m=1}^{\infty}\binom{\frac{1}{2}}{m}Q^m}{(\lambda_1\lambda_3 - y^2)^{1/2}\Lambda^{(d+2)/2}}\left[\frac{y}{\lambda_3}\left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu d}{2D(t_2+\tau)}\right) - \left(\frac{\mathbf{a}^2}{z^2 w_\mu} - \frac{w_\mu d}{2D(t_1+\tau)}\right)\right], \tag{C28}$$

$$T_2 = \frac{(\Lambda-1)(y\mathbf{c}^2 - \lambda_3\mathbf{a}\cdot\mathbf{c})}{\Lambda^{(d+2)/2}\lambda_3(\lambda_1\lambda_3 - y^2)^{1/2}(1-Q)^{1/2}[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]}\left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu d}{2D(t_2+\tau)}\right), \tag{C29}$$

$$T_3 = \frac{2z(\lambda_3\mathbf{a}\cdot\mathbf{c} - y\mathbf{c}^2)}{\Lambda^{d/2}(\lambda_1\lambda_3 - y^2)^{1/2}(1-Q)^{1/2}[\Lambda z\lambda_3 - (\Lambda-1)\mathbf{c}^2]^2}\left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu}{2D(t_2+\tau)}\right), \tag{C30}$$

$$T_4 = \frac{z^{-4}(\mathbf{a}^2\mathbf{c}^2 - \mathbf{a}\cdot\mathbf{c}^2)}{\Lambda^{(d+6)/2}(\lambda_1\lambda_3 - y^2)^{3/2}(1-Q)^{3/2}}\left(\frac{[\Lambda zy - (\Lambda-1)\mathbf{a}\cdot\mathbf{c}]}{v_\mu} - \frac{[\Lambda z\lambda_1 - (\Lambda-1)\mathbf{a}^2]}{w_\mu}\right), \tag{C31}$$

$$T_5 = \frac{-[\lambda_3\mathbf{a}^2 - 2y\mathbf{a}\cdot\mathbf{c} + \lambda_1\mathbf{c}^2]}{\Lambda^{(d+4)/2}z(\lambda_1\lambda_3 - y^2)^{3/2}(1-Q)^{3/2}}\left[\frac{yR}{\lambda_3}\left(\frac{\mathbf{c}^2}{z^2 v_\mu} - \frac{v_\mu}{2D(t_2+\tau)}\right) - \left(\frac{\mathbf{a}^2}{z^2 w_\mu} - \frac{w_\mu}{2D(t_1+\tau)}\right)\right], \tag{C32}$$

$$T_6=\frac{2(\Lambda-1)(\mathbf{a}^2\mathbf{c}^2-\mathbf{a}\cdot\mathbf{c}^2)}{\Lambda^{(d+6)/2}z^2(\lambda_1\lambda_3-y^2)^{3/2}(1-Q)^{3/2}}\left(\frac{yR}{\lambda_3}\left(\frac{\mathbf{c}^2}{z^2v_\mu}-\frac{v_\mu}{2D(t_2+\tau)}\right)-\left(\frac{\mathbf{a}^2}{z^2w_\mu}-\frac{w_\mu}{2D(t_1+\tau)}\right)\right]. \tag{C33}$$

APPENDIX D

1. Large-d behavior of T_n

In this section we will demonstrate that the large-d behavior of the terms T_i, which are defined by Eqs. (C28)–(C33), is controlled, and hence that we are justified in applying the method of steepest descents to the $\hat{\mu}$ and ψ integrals in Eq. (71). To simplify the algebra within this section we define several new variables:

$$f_1=(1-x/d),\quad \gamma=\left(\frac{4t_1t_2}{(t_1+t_2)^2}\right)^{d/4}\exp\left(\frac{-r^2}{4D(t_1+t_2)}\right),$$

$$f_2=(1-x/2d),\quad E=f_2{}^{(-d+2)/2}\exp\left(-\frac{x\hat{\boldsymbol{\mu}}^2}{8Df_2}\right),$$

$$f_3=\left(1-\frac{t_2}{t_1+t_2}x/d\right),\quad E_s=f_3{}^{(-d+2)/2}\exp\left(\frac{\mathbf{r}^2}{4D(t_1+t_2)}-\frac{[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2}{4D(t_1+t_2)f_3}\right).$$

Therefore, the set of variables given by Eqs. (A8), with which the terms T_i are defined, may be rewritten as

$$z=4Dt_2f_1,\quad w_\mu=\gamma f_3f_1^{d/2}\left(\frac{t_2}{t_1}\right)^{d/4}E_s,$$

$$\lambda_1=\left(\frac{t_2}{t_1}\right)^{d/2}f_1^{d/2},\quad y=\lambda\left(\frac{t_2}{t_1}\right)^{d/4}f_1^{d/2}\gamma, \tag{D1}$$

$$\lambda_3=f_1^{d/2},\quad \mathbf{a}=\frac{2t_2\gamma}{t_1+t_2}f_1^{(d+2)/2}\left(\frac{t_2}{t_1}\right)^{d/4}[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]E_s,$$

$$v_\mu=f_2f_1^{d/2}E,\quad \mathbf{c}=-(xt_2)^{1/2}f_1^{(d+2)/2}\hat{\boldsymbol{\mu}}E.$$

To investigate the large-d behavior of the T_n, we substitute these variables back into Eqs. (C28)–(C33) to obtain

$$T_1=\frac{\gamma\sum_{m=1}^{\infty}\binom{\frac{1}{2}}{m}Q^m}{2D\Lambda^{(d+2)/2}(1-\gamma^2)^{1/2}}\left[\frac{E}{2t_2}\left(\frac{x\hat{\boldsymbol{\mu}}^2}{4Df_2}-d\right)-\frac{E_s}{t_1+t_2}\left(\frac{[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2}{2Df_3(t_1+t_2)}-d\right)\right], \tag{D2}$$

$$T_2=\frac{-\gamma(\Lambda-1)f_1^{(d+2)/2}E^2}{4Dt_2(t_1+t_2)\Lambda^{(d+2)/2}(1-\gamma^2)^{1/2}(1-Q)^{1/2}}\left(\frac{x\hat{\boldsymbol{\mu}}^2}{4Df_2}-d\right)\frac{\{x\hat{\boldsymbol{\mu}}^2[2t_2E_s-(t_1+t_2)E]-2(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}E_s\}}{4D\Lambda-(\Lambda-1)f_1^{(d+2)/2}x\hat{\boldsymbol{\mu}}^2E^2}, \tag{D3}$$

$$T_3=\frac{2\gamma f_1^{(d+2)/2}E^2}{\Lambda^{d/2}t_2(t_1+t_2)(1-\gamma^2)^{1/2}(1-Q)^{1/2}}\left(\frac{x\hat{\boldsymbol{\mu}}^2}{4Df_2}-1\right)\frac{\{x\hat{\boldsymbol{\mu}}^2[2t_2E_s-(t_1+t_2)E]-2(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}E_s\}}{[4D\Lambda-(\Lambda-1)f_1^{(d+2)/2}x\hat{\boldsymbol{\mu}}^2E^2]^2}, \tag{D4}$$

$$T_4=\frac{xf_1^{(d+2)/2}EE_s(\hat{\boldsymbol{\mu}}^2\mathbf{r}^2-\hat{\boldsymbol{\mu}}\cdot\mathbf{r}^2)}{16D^4\Lambda^{(d+6)/2}(t_1+t_2)^2(1-\gamma^2)^{3/2}(1-Q)^{3/2}}\left[\frac{\gamma^3E_s}{f_2}\left(D\Lambda-\frac{(\Lambda-1)f_1^{(d+2)/2}EE_s}{2(t_1+t_2)}[xt_2\hat{\boldsymbol{\mu}}^2-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}]\right)\right.$$
$$\left.-\frac{\gamma E}{f_3}\left(D\Lambda-\frac{\gamma^2(\Lambda-1)t_2f_1^{(d+2)/2}E_s^2}{(t_1+t_2)^2}[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2\right)\right], \tag{D5}$$

$$T_5=\frac{-\gamma(1-\gamma^2)^{-3/2}f_1^{(d+2)/2}}{8D^2\Lambda^{(d+4)/2}(1-Q)^{3/2}}\left[\frac{RE}{2t_2}\left(\frac{x\hat{\boldsymbol{\mu}}^2}{4Df_2}-1\right)-\frac{E_s}{t_1+t_2}\left(\frac{[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2}{2D(t_1+t_2)f_3}-1\right)\right]$$
$$\times\left[xE^2\hat{\boldsymbol{\mu}}^2+\frac{4t_2\gamma^2E_s^2}{(t_1+t_2)^2}(\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2-\frac{4\gamma^2EE_s}{(t_1+t_2)}[xt_2\hat{\boldsymbol{\mu}}^2-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}]\right], \tag{D6}$$

$$T_6=\frac{(\Lambda-1)xt_2\gamma^3f_1^{d+2}E^2E_s^2(\hat{\boldsymbol{\mu}}^2\mathbf{r}^2-\hat{\boldsymbol{\mu}}\cdot\mathbf{r}^2)}{4D^3(t_1+t_2)^2\Lambda^{(d+6)/2}(1-\gamma^2)^{3/2}(1-Q)^{3/2}}\left[\frac{RE}{2t_2}\left(\frac{x\hat{\boldsymbol{\mu}}^2}{4Df_2}-1\right)-\frac{E_s}{(t_1+t_2)}\left(\frac{[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2}{2D(t_1+t_2)f_3}-1\right)\right], \tag{D7}$$

where

$$Q=\frac{(\Lambda-1)f_1^{(d+2)/2}}{4\Lambda D(1-\gamma^2)}\left[\frac{4t_2\gamma^2E_s^2}{(t_1+t_2)^2}[\mathbf{r}-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}]^2-\frac{4\gamma^2EE_s}{(t_1+t_2)}[xt_2\hat{\boldsymbol{\mu}}^2-(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}]+xE^2\hat{\boldsymbol{\mu}}^2\right.$$
$$\left.-\frac{xt_2(\Lambda-1)f_1^{(d+2)/2}}{\Lambda D}\left(\frac{\gamma EE_s}{t_1+t_2}\right)^2(\boldsymbol{\mu}^2\mathbf{r}^2-\hat{\boldsymbol{\mu}}\cdot\mathbf{r}^2)\right], \tag{D8}$$

$$R=1+\frac{(\Lambda-1)Ef_1^{(d+2)/2}\{x\hat{\boldsymbol{\mu}}^2[(t_1+t_2)E-2t_2E_s]+2E_s(xt_2)^{1/2}\hat{\boldsymbol{\mu}}\cdot\mathbf{r}\}}{(t_1+t_2)[4D\Lambda-(\Lambda-1)f_1^{(d+2)/2}x\hat{\boldsymbol{\mu}}^2E^2]}. \tag{D9}$$

$\Lambda=1+2\tilde{w}/d$, and $\binom{\frac{1}{2}}{m}$ are binomial coefficients.

Although the above expressions are rather complicated, we are only interested in determining whether the large-d behavior in each case is bounded. We notice that all the factors $(t_2/t_1)^{d/4}$ cancel, and the only terms which have a d-dependent exponent are $f_1^{d/2}$, $f_2^{d/2}$, $f_3^{d/2}$ and $\Lambda^{d/2}$. However, the large-d limit of each of these expressions is independent of d. We have

$$\lim_{d\to\infty}f_1^{d/2}=\exp\left(-\frac{x}{2}\right),\quad \lim_{d\to\infty}f_2^{d/2}=\exp\left(-\frac{x}{4}\right), \tag{D10}$$

$$\lim_{d\to\infty}f_3^{d/2}=\exp\left(-\frac{t_2x}{2(t_1+t_2)}\right),\quad \lim_{d\to\infty}\Lambda^{-d/2}=\exp(-\tilde{w}). \tag{D11}$$

At large d therefore, all the terms T_n in Eq. (71), are dominated by the exponential factor $\exp[-dg(x)]$, and hence we may complete the integral using the method of steepest descents.

2. 1/*d* expansion

In this section we evaluate the leading order term in the $1/d$ expansion of each of the expressions for T_n [given by Eqs. (D2)–(D7)] evaluated at $\hat{\boldsymbol{\mu}}^2=2D$ and $\psi=\pi/2$, where ψ is the angle between $\mathbf{r}$ and $\hat{\boldsymbol{\mu}}$. Since the final part of this calculation requires the integration of these terms over the variables x and $\tilde{w}$, it is important to check that these integrals do not alter the d dependence of higher orders in the expansion; this ensures that we have calculated the entire leading order contribution. If we examine the expression for each T_n in turn, we see that, in the large-d limit, every order in the $1/d$ expansion will have an exponential factor with a negative x and $\tilde{w}$ exponent. The presence of these exponential factors ensures that the x and $\tilde{w}$ integrals will not alter the d dependence at any order in the expansion, so we only need to calculate the leading order terms.

We first evaluate the variables E, E_s, R, and Q at the position of the minimum which controls the value of the integral in Eq. (71) ($\hat{\mu}=\sqrt{2D},\psi=\pi/2$); then expanding to leading order in $1/d$ gives $E=1$, $E_s=1$, $R=1$ and

$$Q=\frac{\tilde{w}\,\exp(-x/2)}{d(1-\gamma^2)}\left[x\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)+\frac{2t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right]. \tag{D12}$$

Using these results we can now calculate the terms up to $O(1)$ in the expansion of each expression for T_n [Eqs. (D2)–(D7)]; these are given by

$$T_1=\frac{\gamma(t_2-t_1)\exp(-x/2)\tilde{w}\ \exp(-\tilde{w})}{8Dt_2(t_1+t_2)(1-\gamma^2)^{3/2}}\left[x\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)+\frac{2t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right], \tag{D13}$$

$$T_2=\frac{\gamma(t_2-t_1)x\ \exp(-x/2)\tilde{w}\ \exp(-\tilde{w})}{4Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{D14}$$

$$T_3=\frac{\gamma(t_2-t_1)x(x-2)\exp(-x/2)\exp(-\tilde{w})}{8Dt_2(t_1+t_2)(1-\gamma^2)^{1/2}}, \tag{D15}$$

$$T_4=\frac{-\gamma\mathbf{r}^2x\ \exp(-x/2)\exp(-\tilde{w})}{8D^2(t_1+t_2)^2(1-\gamma^2)^{1/2}}, \tag{D16}$$

$$T_5=\frac{-\gamma\ \exp(-x/2)\exp(-\tilde{w})}{8Dt_2(1-\gamma^2)^{3/2}}\left[x\left(1-\frac{4t_1t_2\gamma^2}{(t_1+t_2)^2}\right)+\frac{2t_2\gamma^2\mathbf{r}^2}{D(t_1+t_2)^2}\right]\left[\frac{t_2-t_1}{t_1+t_2}-\frac{t_2\mathbf{r}^2}{D(t_1+t_2)^2}+\frac{x}{2}\left(1-\frac{4t_2^2}{(t_1+t_2)^2}\right)\right], \tag{D17}$$

$$T_6=O(1/d). \tag{D18}$$

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